Performance Analysis of a Multi-Relay Cooperative Diversity Network with Decode and Forward Relaying

M. D. Selvaraj
Department of Electrical Engineering
Indian Institute of Technology - Delhi
Hauz Khas, New Delhi 110016, India
E-mail: selvarajmd@yahoo.com

Ranjan K. Mallik
Department of Electrical Engineering
Indian Institute of Technology - Delhi
Hauz Khas, New Delhi 110016, India
E-mail: rkmallik@ee.iitd.ernet.in

Abstract — **Consider a multi-relay cooperative diversity network consisting of a source, N relays, and a destination. For the transmission of a data symbol from the source to the destination, the relays cooperate with the source through the decode and forward protocol. All the links are statistically independent and undergo flat Rayleigh fading. Conventional instantaneous signal-to-noise ratio based selection combining at the destination chooses one link from the set consisting of the source-to-destination link and the N relay-to-destination links. However, this scheme does not account for the effect of the source-to-relay links in the error performance. To overcome this drawback, we present here a scaled selection combining scheme, which accounts for this effect through a deterministic scale factor. The end-to-end symbol error probability (SEP) of this scheme with binary phase-shift keying is derived in closed form. We also give a method of obtaining the optimum scale factor that minimizes the end-to-end SEP.**

Keywords – **Binary phase-shift keying (BPSK), decode and forward (DF) protocol, Rayleigh fading, symbol error probability (SEP), scaled selection combining.**

I. Introduction

Spatial diversity is a well known technique for mitigating fading. User cooperation by means of relaying through the decode and forward (DF) protocol is a recently proposed method [1, 2] for achieving space diversity without employing multiple antennas at the mobile units. In [3], the diversity order analysis with relay selection of cooperative maximal-ratio combining and link-adaptive regeneration schemes is investigated. The outage and error probabilities of threshold based opportunistic relaying and selection cooperation schemes have been investigated in [4]. In [5], the performance of amplify and forward relaying with selection combining of single relay cooperative diversity has been analyzed. A relay selection scheme, which utilizes the partial channel state information (CSI) available at the source and the relays for achieving higher bandwidth efficiency and full diversity order, has been presented in [6]. The outage analysis of a cooperative diversity system with opportunistic DF relaying has been analyzed in [7]. In [8], a paired error approach has been used to analyze the error performance of a cooperative diversity system.

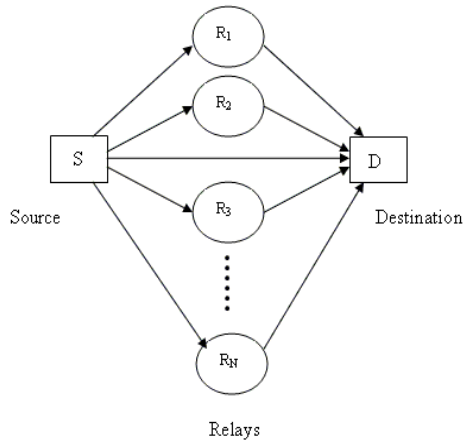

Fig. 1. Cooperative diversity network

In this paper, we consider a multi-relay cooperative diversity network consisting of a source, N relays, and a destination. For the transmission of a data symbol from the source to the destination, the relays cooperate with the source through the DF protocol. An implementable low complexity receiver at the destination uses selection combining by choosing one link from the set consisting of the source-to-destination link and the N relay-to-destination links. However, conventional instantaneous signal-to-noise ratio (SNR) based selection combining does not account for the effect of the source-to-relay links in the error performance. To overcome this drawback, we present here a scaled selection combining scheme, which accounts for this effect through a deterministic scale factor. For a flat Rayleigh fading environment with statistically independent links, the end-to-end symbol error probability (SEP) of this scheme with binary phase-shift keying (BPSK) is derived in closed form. We also give a method of obtaining the optimum scale factor that minimizes the end-to-end SEP.

II. System Model

Consider a cooperative diversity network as shown in Fig. 1, which has a source S transmitting to a destination D through N cooperating relays $\mathrm{R}_1, \ldots, \mathrm{R}_N$. The relays cooperate with the source through the DF protocol. All the links are mutually independent and subject to flat Rayleigh fading.

There are two phases of operation. In phase I, the source broadcasts its data to the destination and the N relays. The complex baseband received signal at the destination and at the ith relay, are modeled, respectively, as

$$r_{sd} = h_{sd} s + n_{sd}\,, \tag{1}$$

$$r_{sr_i} = h_{sr_i} s + n_{sr_i}\,, \quad i = 1, \ldots, N, \tag{2}$$

where s is the information-bearing BPSK symbol having energy $2E_s$ and is given by

$$s \in \{-\sqrt{2E_s}, \sqrt{2E_s}\}\,, \tag{3}$$

h_{sd}, h_{sr_i} are the complex fading gains and n_{sd}, n_{sr_i} the additive noises of the source-to-destination link and the source-to-ith-relay link, respectively. The fading gains h_{sd}, $h_{sr_1}, \ldots, h_{sr_N}$ are mutually independent and are modeled as zero-mean complex circular Gaussian random variables with h_{sd} having variance Ω_{sd} and each h_{sr_i} having variance Ω_{sr} for $i = 1, \ldots, N$. The additive noises n_{sd}, $n_{sr_1}, \ldots, n_{sr_N}$, which are independent of the fading gains, are also mutually independent zero-mean complex circular Gaussian random variables, each having variance $2N_0$.

In phase II, each relay detects the source data and forwards the detected data $\hat{s}$ to the destination. The complex baseband signal received from the ith relay at the destination is modeled as

$$r_{rd_i} = h_{rd_i} \hat{s} + n_{rd_i}\,, \quad i = 1, \ldots, N, \tag{4}$$

where h_{rd_i} and n_{rd_i} are the complex fading gain and the additive noise, respectively, of the ith-relay-to-destination link. The quantities $h_{rd_1}, \ldots, h_{rd_N}$ and $n_{rd_1}, \ldots, n_{rd_N}$ are mutually independent zero-mean complex circular Gaussian random variables that are also independent of h_{sd}, n_{sd}, $h_{sr_1}, \ldots, h_{sr_N}$, $n_{sr_1}, \ldots, n_{sr_N}$, with each h_{rd_i} having variance Ω_{rd} and each n_{rd_i} having variance $2N_0$ for $i = 1, \ldots, N$.

We assume that the ith relay has the CSI of the source-to-ith-relay link for $i = 1, \ldots, N$, and the destination has the CSI of the source-to-destination link and the N relay-to-destination links.

The detected symbol $\hat{s}$ resulting from coherent detection at the ith relay is obtained from the decision rule

$$\hat{s} = \sqrt{2E_s}\,\mathrm{sgn}\left(\mathrm{Re}\left(h^*_{sr_i} r_{sr_i}\right)\right)\,, \tag{5}$$

where $\mathrm{sgn}(\cdot)$ denotes the signum or sign function and $(\cdot)^*$ the complex conjugate.

Let the *instantaneous SNR* of the source-to-destination link, the source-to-ith-relay link, and the ith-relay-to-destination link be denoted as

$$\gamma_{sd} = \frac{E_s}{N_0}|h_{sd}|^2\,, \;\; \gamma_{sr_i} = \frac{E_s}{N_0}|h_{sr_i}|^2\,, \;\; \gamma_{rd_i} = \frac{E_s}{N_0}|h_{rd_i}|^2\,, \tag{6}$$

and the corresponding *average SNRs* be denoted as

$$\Gamma_{sd} = \mathbf{E}[\gamma_{sd}] = \frac{E_s \Omega_{sd}}{N_0}\,,$$

$$\Gamma_{sr} = \mathbf{E}[\gamma_{sr_i}] = \frac{E_s \Omega_{sr}}{N_0}\,, \;\; \Gamma_{rd} = \mathbf{E}[\gamma_{rd_i}] = \frac{E_s \Omega_{rd}}{N_0}\,, \tag{7}$$

where $\mathbf{E}[\cdot]$ is the expectation operator. Thus γ_{sd}, γ_{sr_i}, and γ_{rd_i} are exponentially distributed with means Γ_{sd}, Γ_{sr}, and Γ_{rd}, respectively.

At the destination, we choose the source-to-destination link if

$$\gamma_{sd} = \max\{\gamma_{sd}, \beta\gamma_{rd_1}, \ldots, \beta\gamma_{rd_N}\}\,,$$

where β is a *deterministic positive scale factor*, otherwise, if

$$\beta\gamma_{rd_i} = \max\{\gamma_{sd}, \beta\gamma_{rd_1}, \ldots, \beta\gamma_{rd_N}\}\,,$$

we choose the ith-relay-to-destination link. The final demodulated symbol $\hat{\hat{s}}$ resulting from scaled selection combining at the destination is obtained from the decision rule

$$\hat{\hat{s}} = \begin{cases} \sqrt{2E_s}\,\mathrm{sgn}\left(\mathrm{Re}\left(h^*_{sd} r_{sd}\right)\right) \\ \qquad \text{if } \; \gamma_{sd} > \beta\gamma_{rd_j}, \;\; j = 1, \ldots, N, \\ \\ \sqrt{2E_s}\,\mathrm{sgn}\left(\mathrm{Re}\left(h^*_{rd_i} r_{rd_i}\right)\right) \\ \qquad \text{if } \; \gamma_{sd} < \beta\gamma_{rd_i} \;\; \text{and} \\ \qquad\quad \beta\gamma_{rd_j} < \beta\gamma_{rd_i}, \;\; j \neq i, \;\; j = 1, \ldots, N, \\ \text{for } \; i = 1, \ldots, N. \end{cases} \tag{8}$$

III. Performance Analysis

We now derive SEP expressions for the source-to-destination and the source-to-ith-relay-to-destination links, and apply them to obtain an expression for the end-to-end SEP. Further, we optimize the scale factor β such that the end-to-end SEP is minimized.

A. Source-to-Destination Link

The conditional SEP of the source-to-destination link, conditioned on γ_{sd}, is given by

$$P_{e_0}(\gamma_{sd}) = Q\left(\sqrt{2\gamma_{sd}}\right)\,, \tag{9}$$

where $Q(\cdot)$ denotes the Gaussian Q-function. Averaging (9) over the exponential statistics γ_{sd}, under the condition

that $\gamma_{sd} > \beta\gamma_{rd_j}, \quad j = 1, \ldots, N$, the *SEP of the source-to-destination link*, denoted as P_{e_0}, is given by

$$P_{e_0} = \int_{u=0}^{\infty}\int_{v_1=0}^{u}\cdots\int_{v_N=0}^{u} Q\left(\sqrt{2u}\right)\frac{1}{\Gamma_{sd}}\exp\left(-\frac{u}{\Gamma_{sd}}\right) \times \prod_{j=1}^{N}\frac{1}{\beta\Gamma_{rd}}\exp\left(-\frac{v_j}{\beta\Gamma_{rd}}\right)dv_1dv_2\cdots dv_N du\,. \quad (10)$$

Simplifying (10), we get,

$$P_{e_0} = \int_0^{\infty} Q\left(\sqrt{2u}\right)\frac{1}{\Gamma_{sd}}e^{-\frac{u}{\Gamma_{sd}}}\left(1-e^{-\frac{u}{\beta\Gamma_{rd}}}\right)^N du\,. \quad (11)$$

Define the function $\eta(k)$ as

$$\eta(k) \triangleq \frac{\beta\Gamma_{rd} + k\Gamma_{sd}}{\beta\Gamma_{sd}\Gamma_{rd}}\,. \quad (12)$$

Using the binomial expansion of the term $\left(1-e^{-\frac{u}{\beta\Gamma_{rd}}}\right)^N$ in (11), we obtain

$$P_{e_0} = \sum_{k=0}^{N}\binom{N}{k}(-1)^k\frac{1}{\Gamma_{sd}}\int_0^{\infty} Q\left(\sqrt{2u}\right)e^{-\eta(k)u}du\,, \quad (13)$$

where $\eta(k)$ is given by (12). Using integration by parts, we obtain (13) in closed form as

$$P_{e_0} = \frac{1}{2\Gamma_{sd}}\sum_{k=0}^{N}\binom{N}{k}\frac{(-1)^k}{\eta(k)}\left(1-\frac{1}{\sqrt{1+\eta(k)}}\right)\,. \quad (14)$$

B. Source-to-Relay-to-Destination Link

The conditional SEP of the source-to-ith-relay link and the ith-relay-to-destination link, conditioned on the instantaneous SNRs γ_{sr_i} and γ_{rd_i}, respectively, are expressed as

$$P_{esr_i}(\gamma_{sr_i}) = Q\left(\sqrt{2\gamma_{sr_i}}\right)\,, \quad (15)$$

$$P_{erd_i}(\gamma_{rd_i}) = Q\left(\sqrt{2\gamma_{rd_i}}\right)\,. \quad (16)$$

The conditional SEP of the source-to-ith-relay-to-destination link, conditioned on γ_{sr_i} and γ_{rd_i}, is given by

$$P_{e_i}(\gamma_{sr_i},\gamma_{rd_i}) = Q\left(\sqrt{2\gamma_{sr_i}}\right)\left(1-Q\left(\sqrt{2\gamma_{rd_i}}\right)\right) + \left(1-Q\left(\sqrt{2\gamma_{sr_i}}\right)\right)Q\left(\sqrt{2\gamma_{rd_i}}\right)\,. \quad (17)$$

Define the quantity ρ_{sr} as

$$\rho_{sr} \triangleq \sqrt{\frac{\Gamma_{sr}}{1+\Gamma_{sr}}}\,. \quad (18)$$

Averaging (17) over the exponential statistics γ_{sr_i}, we get

$$P_{e_i}(\Gamma_{sr},\gamma_{rd_i}) = \frac{1}{2}(1-\rho_{sr}) + \rho_{sr}Q\left(\sqrt{2\gamma_{rd_i}}\right)\,. \quad (19)$$

Now we average (19) over the statistics of γ_{rd_i} under the conditions $\gamma_{sd} < \beta\gamma_{rd_i}, \quad \beta\gamma_{rd_j} < \beta\gamma_{rd_i}, \quad j \neq i$, $j = 1, \ldots, N$. This results in the *SEP of the source-to-ith-relay-to-destination link*, denoted as P_{e_i}, being expressed as

$$P_{e_i} = \int_{v_i=0}^{\infty}\int_{u=0}^{v_i}\int_{v_1=0}^{v_i}\cdots\int_{v_N=0}^{v_i}\left[\frac{1}{2}(1-\rho_{sr}) + \rho_{sr}Q\left(\sqrt{\frac{2v_i}{\beta}}\right)\right]\left[\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{\beta\Gamma_{rd}}e^{-\frac{v_j}{\beta\Gamma_{rd}}}\right] \times\frac{1}{\Gamma_{sd}}e^{-\frac{u}{\Gamma_{sd}}}\frac{1}{\beta\Gamma_{rd}}e^{-\frac{v_i}{\beta\Gamma_{rd}}}dv_1\cdots dv_N du dv_i\,. \quad (20)$$

Simplifying (20), we obtain

$$P_{e_i} = \int_0^{\infty}\left(1-e^{-\frac{v_i}{\Gamma_{sd}}}\right) \times\left[\frac{1}{2}(1-\rho_{sr}) + \rho_{sr}Q\left(\sqrt{\frac{2v_i}{\beta}}\right)\right] \times\frac{1}{\beta\Gamma_{rd}}e^{-\frac{v_i}{\beta\Gamma_{rd}}}\left(1-e^{-\frac{v_i}{\beta\Gamma_{rd}}}\right)^{N-1}dv_i\,. \quad (21)$$

Using the binomial expansion of the term $\left(1-e^{-\frac{v_i}{\beta\Gamma_{rd}}}\right)^{N-1}$ in (21), we get

$$P_{e_i} = \sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\int_0^{\infty}\left(1-e^{-\frac{v_i}{\Gamma_{sd}}}\right) \times\left[\frac{1}{2}(1-\rho_{sr}) + \rho_{sr}Q\left(\sqrt{\frac{2v_i}{\beta}}\right)\right] \times\frac{1}{\beta\Gamma_{rd}}e^{-\frac{(k+1)v_i}{\beta\Gamma_{rd}}}dv_i\,. \quad (22)$$

We express (22) in closed form using integration by parts as

$$P_{e_i} = \sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k \times\left[\frac{1}{2(1+k)}\left(1-\rho_{sr}\sqrt{\frac{\Gamma_{rd}}{1+k+\Gamma_{rd}}}\right) + \frac{1}{2(1+\eta(k)\beta\Gamma_{rd})} \times\left(\rho_{sr}\sqrt{\frac{\Gamma_{rd}}{1+\Gamma_{rd}+\eta(k)\beta\Gamma_{rd}}}-1\right)\right]\,. \quad (23)$$

Note that owing to symmetry among the relays, P_{e_i} *is independent of* i.

C. End-to-End SEP

Combining the source-to-destination link SEP P_{e_0} given by (14) and the source-to-relay-to-destination link SEP P_{e_i} given by (23), we obtain the *end-to-end SEP* P_e *of the scaled selection combining scheme* from

$$P_e = P_{e_0} + NP_{e_i}\,, \tag{24}$$

which is expressed in closed form as

$$\begin{aligned} P_e \;=\; & \frac{1}{2\Gamma_{sd}}\sum_{k=0}^{N}\binom{N}{k}\frac{(-1)^k}{\eta(k)}\left(1-\frac{1}{\sqrt{1+\eta(k)}}\right) \\ & +N\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k \\ & \times\left[\frac{1}{2(1+k)}\left(1-\rho_{sr}\sqrt{\frac{\Gamma_{rd}}{1+k+\Gamma_{rd}}}\right)\right. \\ & +\frac{1}{2(1+\eta(k)\beta\Gamma_{rd})} \\ & \left.\times\left(\rho_{sr}\sqrt{\frac{\Gamma_{rd}}{1+\Gamma_{rd}+\eta(k)\beta\Gamma_{rd}}}-1\right)\right], \end{aligned} \tag{25}$$

where $\eta(k)$ is given by (12) and ρ_{sr} by (18).

D. Optimizing the Scale Factor β

We minimize the end-to-end error probability given by (25) by optimizing the scale factor β. Differentiating the right-hand side of (25) with respect to β and putting the result to zero, we obtain the equation

$$\begin{aligned} & \sqrt{\beta}\sum_{k=1}^{N}\binom{N-1}{k-1}(-1)^{k-1} \\ & \times\frac{(3k\Gamma_{sd}+3\beta\Gamma_{rd}+2\beta\Gamma_{sd}\Gamma_{rd})}{(k\Gamma_{sd}+\beta\Gamma_{rd})^2\,(k\Gamma_{sd}+\beta\Gamma_{rd}+\beta\Gamma_{sd}\Gamma_{rd})^{3/2}} \\ = \;\; & \rho_{sr}\sum_{k=1}^{N}\binom{N-1}{k-1}(-1)^{k-1} \\ & \times\frac{(3k\Gamma_{sd}+3\beta\Gamma_{rd}+2\Gamma_{sd}\Gamma_{rd})}{(k\Gamma_{sd}+\beta\Gamma_{rd})^2\,(k\Gamma_{sd}+\beta\Gamma_{rd}+\Gamma_{sd}\Gamma_{rd})^{3/2}}\,. \end{aligned} \tag{26}$$

By numerically solving (26) for β, we obtain a value of β for which the SEP is minimum, which we call the *optimum scale factor* β_{opt}. A system designer should select $\beta = \beta_{opt}$ for the best performance of the scheme. We also observe from (26) that when the average SNR of the source-to-relay link Γ_{sr} is large, that is, $\rho_{sr} \approx 1$ (see (18)), the optimum value of β which satisfies (26) is given by $\beta_{opt} \approx 1$.

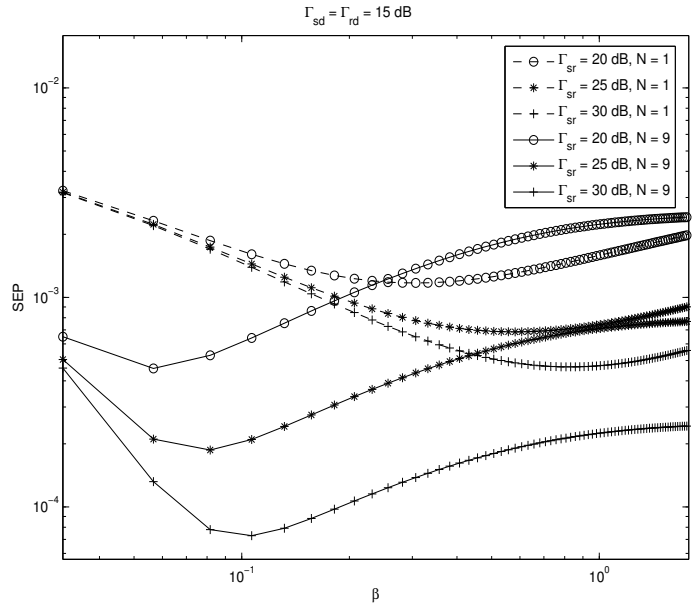

Fig. 2. SEP versus scale factor β with varying average SNR of source-to-relay link Γ_{sr} and number of relays N, average SNR of source-to-destination link $\Gamma_{sd} = 15$ dB, average SNR of relay-to-destination link $\Gamma_{rd} = 15$ dB

IV. Numerical Results

Plots of the SEP P_e versus the scale factor β with varying average SNR of the source-to-relay link Γ_{sr} and number of cooperating relays N, average SNR of the source-to-destination link $\Gamma_{sd} = 15$ dB, average SNR of the relay-to-destination link $\Gamma_{rd} = 15$ dB, computed using (25), are shown in Fig. 2. It is found that the SEP varies with β and reaches a minimum at $\beta = \beta_{opt}$. We also observe that the value of β_{opt} is between zero and unity. As Γ_{sr} increases, β_{opt} increases, and approaches unity as Γ_{sr} becomes large. Thus the scale factor incorporates the effect of the source-to-relay links. Further, as N increases, β_{opt} as well as the SEP at $\beta = \beta_{opt}$ decrease.

Plots of β_{opt} versus Γ_{sr} with different values of Γ_{sd}, Γ_{rd}, and N are shown in Fig. 3. We find that β_{opt} is sensitive to the variation of Γ_{sr}, and the sensitivity decreases with increase of N. It is also observed that β_{opt} is more sensitive to the variation of Γ_{rd} than to the variation of Γ_{sd}.

Plots of the SEP using $\beta = \beta_{opt}$ versus Γ_{sd} (with $\Gamma_{sd} = \Gamma_{rd}$) for different values of Γ_{sr} and N are shown in Fig. 4. The computed SEP values, obtained from (25), and the SEP values obtained by simulation match well, implying the validation of our analysis. We observe that the SEP decreases as Γ_{sr} increases, and tends to reach saturation as Γ_{sr} becomes large. We also find that when more relays cooperate, the error performance improves.

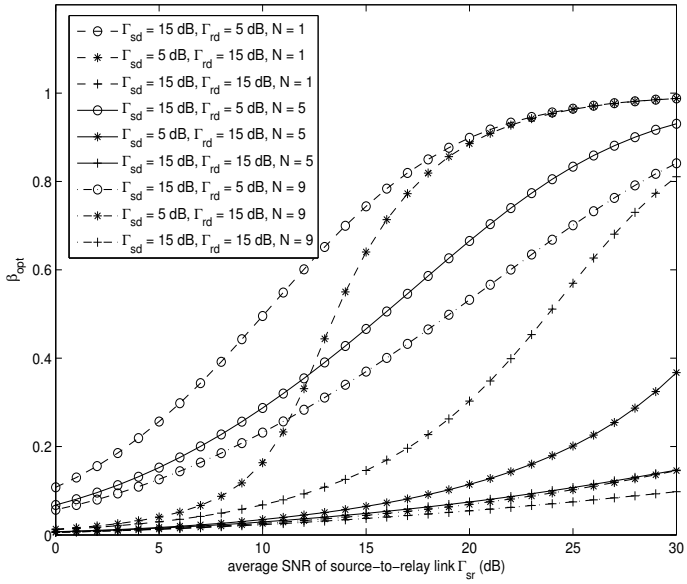

Fig. 3. Optimum scale factor β_{opt} versus average SNR of source-to-relay link Γ_{sr} with varying Γ_{sd}, Γ_{rd}, and N

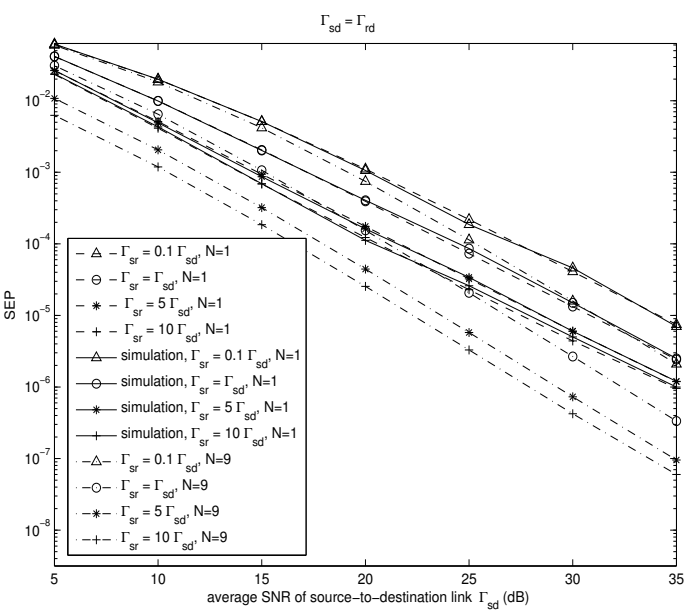

Fig. 4. SEP versus average SNR of source-to-destination link Γ_{sd} ($\Gamma_{sd} = \Gamma_{rd}$) with varying Γ_{sr} and N

V. Conclusion

The conventional instantaneous SNR based selection combining scheme does not consider the effect of the source-to-relay links in the error performance in a multi-relay cooperative diversity scenario. Hence we propose a scaled selection combining scheme that uses a deterministic scale factor to account for this effect. For flat Rayleigh fading, the SEP of this scheme with BPSK is derived in closed form. The optimum scale factor β_{opt} that minimizes the SEP is numerically obtained. It is found that $0 < \beta_{opt} < 1$, and that β_{opt} increases with increase in the average SNR of the source-to-relay link Γ_{sr} and decreases with increase in the number of relays, tending to reach unity as Γ_{sr} becomes large.

References

[1] A. Sendonaris, E. Erkip, and B. Aazhang, "User cooperation diversity - part I: System description," *IEEE Transactions on Communications*, vol. 51, no. 11, pp. 1927–1938, November 2003.

[2] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Transactions on Information Theory*, vol. 50, no. 12, pp. 3062–3080, December 2004.

[3] Z. Yi and I.-M. Kim " Diversity order analysis of the decode-and-forward cooperative networks with relay selection ," *IEEE Transactions on Wireless Communications*, vol. 7, no. 5, pp. 1792–1799, May 2008.

[4] D. S. Michalopoulos and G. K. Karagiannidis, "Performance analysis of single relay selection in Rayleigh fading," *IEEE Transactions on Wireless Communications*, vol. 7, no. 10, pp. 3718–3724, October 2008.

[5] S. Ikki, and M. H. Ahmed "Performance of selection combining in cooperative relaying networks over Rayleigh fading channel," in *Conference Record of the IEEE Canadian Conference on Electrical and Computer Engineering*, May 2008, pp. 831–836.

[6] A. S. Ibrahim, A. K. Sadek, W. Su, and K. J. R. Liu, "Cooperative Communications with Relay-Selection: When to Cooperate and Whom to Cooperate With?," *IEEE Transactions on Wireless Communications*, vol. 7, no. 7, pp. 2814–2827, July 2008.

[7] K.-S. Hwang, Y.-C. Ko, and M.-S. Alouini, "Outage probability of cooperative diversity systems with opportunistic relaying based on decode-and-forward," *IEEE Transactions on Wireless Communications*, vol. 7, no. 12, pp. 5100–5107, December 2008.

[8] M. D. Selvaraj and Ranjan K. Mallik, "Error analysis of the decode and forward protocol with selection combining," *IEEE Transactions on Wireless Communications*, vol. 8, no. 6, pp. 3086–3094, June 2009.