Diversity Multiplexing Tradeoff of SIMO Maximal-Ratio Combining Scheme

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***Abstract*—The Diversity Multiplexing Tradeoff (DMT) [1] is a compact, yet elegant framework to capture the performance of wireless communication systems at asymptotically high signal to noise ratios (SNR). Recent application of DMT at finite SNR [2] makes the DMT a versatile framework. We have analyzed the DMT for the rate-adaptive SIMO maximal-ratio combining (SIMO-MRC) for asymptotic and non-asymptotic signal to noise ratios. This has resulted in arriving at closed form expression of the diversity gain. The DMT arrived by analysis using the non-asymptotic finite SNR framework is found to be consistent with the known interpretations arrived through the asymptotic high-SNR framework [1]. The results of this analysis have been presented in this paper.**

I. Introduction

A. Diversity Multiplexing Tradeoff

With increasing demand for data rates in wireless communication systems, multiple antennas play a significant role. Though multiple antennas provide diversity as well as multiplexing gain, Zheng and Tse [1] provide a fundamental tradeoff between these, that any coding scheme can achieve. This is the diversity and multiplexing tradeoff (DMT) framework proposed in [1] at asymptotically high signal to noise ratios.

We provide a brief overview of the DMT of a MIMO channel. The quasi-static, frequency flat MIMO channel, with N transmit and M receive antennas is described by

$$\mathbf{Y} = \sqrt{\frac{SNR}{N}}\mathbf{HX} + \mathbf{W} \tag{1}$$

Each of the terms in Eq. (1) is a matrix, $\mathbf{Y}$ and $\mathbf{W}$ is $M \times T$ whereas $\mathbf{H}$ is $M \times N$ and $\mathbf{X}$ is $N \times T$. The quasi-static interval T indicates the coding block length. $\mathbf{H}$ and $\mathbf{W}$ have independent entries from a complex Gaussian distribution, $[h_{i,j}] \sim \mathcal{CN}(0,1), [w_{i,j}] \sim \mathcal{CN}(0,1)$. Assuming a scheme of codes which has rate increasing with signal to noise ratio (SNR), the diversity (d) is defined as the exponent of the average probability of error (P_e) curve [1]. The spatial multiplexing gain (r) provides the degrees of freedom. Here R is the data rate which increases with SNR.

$$\begin{aligned} d &= -\lim_{SNR \longrightarrow \infty} \frac{\log(P_e(SNR))}{\log(SNR)} \\ r &= \lim_{SNR \longrightarrow \infty} \frac{\log(R(SNR))}{\log(SNR)} \end{aligned} \tag{2}$$

For each r, $d^*(r)$ is defined to be the supremum of the diversity advantage achieved over all schemes. Using these definitions and under the condition $T \geq M + N - 1$, the optimal curve $d^*(r)$ for the above MIMO system, is a piece-wise linear function connecting the points $(k, d^*(r)), k = 0, 1, .., \min\{M, N\}$ where

$$d^*(k) = (M-k)(N-k) \tag{3}$$

Eq. (3) provides the optimal DMT curve for the MIMO system described above. The optimal DMT curve changes with the change in distribution (pdf) of the channel [8], and also depends on the asymptotic or nonasymptotic nature of signal to noise ratios [3]. In [3] the definitions in Eq. (2) are generalized for finite signal to noise ratios. The multiplexing gain r at finite signal to noise ratio is defined with respect to the capacity of an AWGN channel.

$$r = \frac{R}{\log_2(1+g\rho)} \tag{4}$$

where R is the data rate, ρ is the signal to noise ratio and g is the array gain. The array gain is introduced at the receiver to take care of low signal to noise ratios. Typically g is considered equal to M [3]. The diversity gain $d(r,\rho)$ at rate r and finite signal to noise ratio ρ as defined in [3] is

$$d(r,\rho) = -\rho\frac{\partial}{\partial\rho}\ln P_{out}(r,\rho) \tag{5}$$

B. Receive Diversity Combining Schemes

Receive diversity combining schemes are very well studied in terms of their BER and outage probability performance. Simon et.al. [6] provide an exhaustive study of these schemes. We look at the maximal-ratio combining (MRC) scheme and analyze it using the DMT framework. We consider a transmitter with single transmit antenna and multiple receive antennas (SIMO) and perform MRC at the receiver. The SIMO system is shown in Fig. 1. The signal being received at all the M antennas is combined coherently with suitable gains. This is the MRC receiver which is shown in Fig. 2. The antenna gain selection is detailed in [5].

C. Prior Work

[6] provides performance analysis of diversity combining schemes in terms of BER and outage probability. Schemes

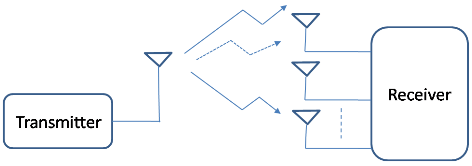

Fig. 1. SIMO system with single transmit and M receive antennas

like MRC, coherent and noncoherent Equal Gain Combining (EGC), Selection Combining (SC), switched and hybrid diversity have been analyzed. Impact of fading correlation and effect of channel estimation error is also considered. BER and Outage Probability expressions for MRC have been derived.

Asymptotic high signal to noise ratio DMT framework has been used to analyze performance of MIMO systems, in particular, orthogonal designs of space-time codes (e.g. Alamouti code), V-BLAST and D-BLAST schemes [1]. [2] analyzes the orthogonal space time block codes (OSTBC) and spatial-multiplexing horizontal encoding (SM-HE) at finite signal to noise ratios. The DMT of OSTBCs at asymptotically high signal to noise ratios as well as finite SNRs is analyzed in [9] in the presence of correlated Nakagami-m fading.

Our contribution is in the analysis of the DMT of SIMO-MRC diversity combining scheme at asymptotically high SNR as proposed in [1] and also at finite SNR as proposed in [2]. Typically the outage probability of the diversity combining schemes is defined in terms of a threshold SNR which is a fixed value of SNR [5], [6]. This fixed value of SNR depends on the required BER performance. In this paper we take an information theoretic approach of analysis of the outage probability of SIMO-MRC.

This approach is independent of a fixed threshold SNR value and hence more versatile. The paper relates the two approaches adopted by [4] and [5] to define the outage probability and combines them using the DMT framework (see Eq. (9) and (10)). The paper also provides a bird's eye view of the diversity as well as multiplexing gain provided by the SIMO-MRC scheme when compared with the optimal tradeoff.

Notation: SNR denotes an asymptotically high signal to noise ratio. ρ denotes a non-asymptotic finite signal to noise ratio.

II. System Model

A SIMO system with a single transmitter and M receive antennas as in Fig. 1 is considered. The received signal is given as

$$\mathbf{y} = \sqrt{SNR}\;\mathbf{h}\;x + \mathbf{n} = \mathbf{z} + \mathbf{n} \tag{6}$$

In Eq. (6), $\mathbf{y}, \mathbf{h}, \mathbf{n}, \mathbf{z}$ are $M \times 1$ vectors. The channel coefficients on each of the path i are independent with circular symmetric complex Gaussian distribution $h_i \sim \mathcal{CN}(0,1) \in \mathbf{h}$. Noise is independent with circular symmetric complex Gaussian distribution $n_i \sim \mathcal{CN}(0,1) \in \mathbf{n}$. At the receiver, each path i is weighted by coefficients α_i as shown in Fig. 2.

$$\alpha_i = a_i \exp\{-j\theta_i\} \text{ where } \; i = 1, 2...M$$

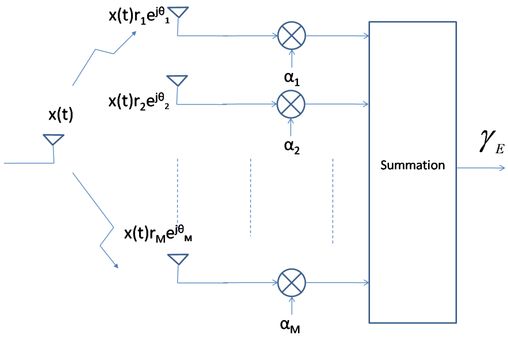

Fig. 2. Linear Combiner for SIMO-MRC

a_i is the weighing coefficient on the i^{th} antenna at the receiver. θ_i is the phase of the incoming signal on the i^{th} antenna at the receiver. The following assumptions are made at the transmitter [1].

- The transmit signal x is normalized to have average transmit power of '1' in symbol. Hence $E|x|^2 = 1$.
- SNR denotes the average signal to noise ratio at each receive antenna.
- A code whose rate increases as $R = r\log(SNR)$ is assumed at the transmitter.

The following assumptions are made at the receiver [5].

- The exact instantaneous channel knowledge is available at the receiver.
- The antenna outputs are co-phased by multiplying each by $\exp\{-j\theta_i\}$.
- The combiner output at the receiver provides the envelope of the received signal which is weighted by a_is.

III. Analysis for SIMO-Maximal Ratio Combining

In MRC, the output of combiner in Fig. 2 is a summation of the signals on all the paths. From Eq. (6), let $r_i = |h_i|$ $i = 1, ...M$

$$y_i = z_i + n_i = \sqrt{SNR}\; r_i\; \exp\{j\theta_i\}\; x + n_i \tag{7}$$

The envelope at the combiner output assuming co-phasing of antennas is

$$z = \sum_i z_i = x\,\sqrt{SNR}\sum_{i=1}^{M} r_i a_i$$

From [5], signal to noise ratio is maximized at the output of the combiner when $a_i^2 = \frac{z_i^2}{N}$ where N is the total noise power.

The SNR at the combiner output is

$$\gamma_E = \frac{1}{N}\sum_{i=1}^{M} z_i^2 = \sum_{i=1}^{M}\gamma_i$$

In the above equation, γ_i represents the signal to noise ratio on each antenna. From [5] the probability distribution function (pdf) of the SNR at the output of the combiner (γ_E) for SIMO-MRC is is

$$p_{\gamma_E}(\gamma) = \frac{\gamma^{M-1}e^{-\gamma/\bar{\gamma}}}{(\bar{\gamma})^M(M-1)!} \tag{8}$$

where $\bar{\gamma}$ is the average branch signal to noise ratio on each branch. This is a Chi-squared distribution with $2M$ degrees of freedom.

A. Outage Probability of SIMO-MRC at asymptotically high signal to noise ratios

The outage probability in terms of a threshold signal to noise ratio γ_o [5] is

$$P_{out}(\gamma_o) = Pr(\gamma_E < \gamma_o) \tag{9}$$

In terms of mutual information ($I(X;Y)$) the outage probability is defined as [4]

$$P_{out}(R) = Pr\big(I(X;Y) < R\big) \tag{10}$$

where R is the data rate. We relate these two definitions of outage probability using the DMT framework. From Fig. 2, at the receiver, the mutual information is given as

$$I(X;Y) = \log(1+\gamma_E) \tag{11}$$

According to the above assumptions, at the transmitter the data rate of the codes increases as $R = r\log(SNR)$.

$$P_{out}(r, SNR) = Pr\Big(\log(1+\gamma_E) < r\log(SNR)\Big)$$

At high SNR,

$$P_{out}(r, SNR) \approx Pr(\gamma_E < SNR^r) \tag{12}$$

From Eq. (8) and (9) and [5], the outage probability of SIMO-MRC is given by

$$P_{out,MRC}(\gamma_o) = 1 - e^{-\frac{\gamma_o}{\bar{\gamma}}}\sum_{k=1}^{M}\frac{(\gamma_o/\bar{\gamma})^{k-1}}{(k-1)!} \tag{13}$$

For a SIMO system, the maximum multiplexing gain from Eq. (3) for $k = 0$ is $r_{max} = \min\{1, M\} = 1$, hence $1 - r \geq 0$. From Eq. (9), (12) and (13), identifying $\gamma_o = SNR^r$ and $\bar{\gamma} = SNR$

$$P_{out,MRC}(r, SNR) \approx 1 - e^{\frac{-1}{SNR^{1-r}}}\sum_{k=1}^{M}\frac{(\frac{1}{SNR^{1-r}})^{k-1}}{(k-1)!} \tag{14}$$

B. DMT of SIMO-MRC at asymptotically high-SNRs

From Eq. (2), diversity gain is defined as

$$d = -\lim_{SNR\to\infty}\frac{\log(P_e(SNR))}{\log(SNR)}$$

According to [1], it is very likely that a detection error occurs when conditioned on the channel outage event. Therefore, the outage probability is a lower bound on the average error probability. Hence in practice, the negative of SNR exponent of $P_{out}(r)$, defined below, is considered as diversity gain.

$$\begin{aligned}
d(r) &= -\lim_{SNR\to\infty}\frac{\log P_{out}(r,SNR)}{\log SNR} && (15)\\
&= -\lim_{SNR\to\infty}\Bigg[\frac{SNR}{P_{out}(r,SNR)} \\
&\qquad \times\frac{\partial P_{out}(r,SNR)}{\partial SNR}\Bigg] \\
&= \lim_{SNR\to\infty}\Bigg[\frac{-SNR(r-1)}{P_{out}(r,SNR)(M-1)!} \\
&\qquad \times\frac{e^{-SNR^{r-1}}}{SNR^{(1+M-rM)}}\Bigg] \\
&= -(r-1)\lim_{SNR\to\infty}(M - SNR^{r-1}) \\
&= -(r-1)\ M - \lim_{SNR\to\infty}\frac{1}{SNR^{1-r}} \\
&= M(1-r) && (16)
\end{aligned}$$

Eq. (16) gives the diversity and multiplexing tradeoff for SIMO-MRC scheme for the asymptotically high-SNRs. This is also the optimal diversity and multiplexing tradeoff for SIMO system for the asymptotically high-SNRs [[7], Chapter 9].

$$d^*(r) = M(1-r) \text{ where } 0 \leq r \leq 1 \tag{17}$$

C. Outage Probability of SIMO-MRC at finite signal to noise ratios

From Eq. (4) and (10) and (11), at finite-SNRs we have,

$$\begin{aligned}
P_{out}(r,\rho) &= Pr\big(\log(1+\gamma_E) < r\log(1+g\rho)\big) \\
P_{out}(r,\rho) &= Pr\big(\gamma_E < (1+g\rho)^r - 1\big) && (18)
\end{aligned}$$

Note that for finite signal to noise ratio case, $\gamma_o = (1+g\rho)^r - 1$ and $\bar{\gamma} = \rho$. In order to simplify the diversity multiplexing gain tradeoff at finite signal to noise ratio, we define another variable

$$\rho_r = \frac{(1+g\rho)^r - 1}{\rho} \tag{19}$$

From Eq. (14) and (18)

$$\begin{aligned}
P_{out,MRC}(r,\rho) &= 1 - \Bigg[e^{-\left(\frac{(1+g\rho)^r-1}{\rho}\right)} \\
&\qquad \sum_{k=1}^{M}\left(\frac{(1+g\rho)^r-1}{\rho}\right)^{k-1}\frac{1}{(k-1)!}\Bigg] \\
&= 1 - \left[e^{-\rho_r}\sum_{k=1}^{M}(\rho_r)^{k-1}\frac{1}{(k-1)!}\right] && (20)
\end{aligned}$$

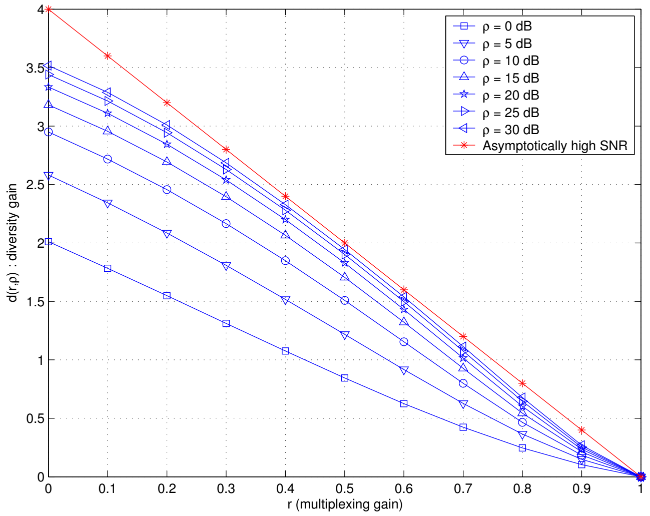

Fig. 3. DMT for SIMO-MRC at finite signal to noise ratios (ρ) for M=4 compared with DMT at asymptotically high-SNR

D. DMT of SIMO-MRC at finite signal to noise ratios

From Eq. (5), (20) and (19) the diversity multiplexing tradeoff at finite signal to noise ratio of SIMO-MRC scheme is given by

$$d(r,\rho) = \left[\frac{-e^{-\rho_r}(\rho_r)^{M-1}}{(1-e^{-\rho_r}\sum_{k=1}^{M}(\rho_r)^{k-1}\frac{1}{(k-1)!})}\right.$$
$$\left.(\frac{1+(1+g\rho)^{r-1}(rg\rho-g\rho-1)}{\rho(M-1)!})\right] \quad (21)$$

1) Finite signal to noise ratio DMT of SIMO-MRC at r=0: From Eq. (19), it is seen that $r = 0$ leads to $\rho_r = 0$. Hence from Eq. (21), it is clear that at $r = 0$ i.e. zero multiplexing gain, leads to an undefined diversity gain at all signal to noise ratios (ρ). In order to derive the value of diversity gain at $r = 0$ and any ρ,

$$\begin{aligned} d(0,\rho) &= \lim_{r\to 0} d(r,\rho) \\ &= \lim_{r\to 0}\left[\frac{-e^{-\rho_r}(\rho_r)^{M-1}}{(1-e^{-\rho_r}\sum_{k=1}^{M}(\rho_r)^{k-1}\frac{1}{(k-1)!})}\right. \\ &\quad \left.(\frac{1+(1+g\rho)^{r-1}(rg\rho-g\rho-1)}{\rho(M-1)!})\right] \\ &= M\left(1-\frac{g\rho}{(1+g\rho)(\log(1+g\rho))}\right) \end{aligned}$$

Eq. (21) is plotted in Fig. 3 for different values of signal to noise ratios ρ and r. This includes the value of diversity gain $d(0,\rho)$. The array gain used is $g = M$ for SIMO-MRC [5].

E. Asymptotic High-SNR DMT from DMT at finite signal to noise ratio

At high-SNR asymptotic values we assume $\rho \to \infty$. Hence it follows that

$$(1+g\rho)^r - 1 \to (1+g\rho)^r$$

$$(\frac{1+g\rho}{\rho})^r \to g^r$$

With $\rho \to \infty$, Eq. (21) can be expressed as

$$\begin{aligned} &\lim_{\rho\to\infty} d(r,\rho) = \\ &= \lim_{\rho\to\infty}\frac{-(r-1)g^{rM}e^{\frac{-(g\rho)^r}{\rho}}\rho^{M(r-1)}}{(1-e^{\frac{-(g\rho)^r}{\rho}}\sum_{k=1}^{M}\frac{(g\rho)^{r(k-1)}}{(\rho^{k-1})(k-1)!})(M-1)!} \\ &= \lim_{\rho\to\infty} -M(r-1)+\frac{g^r(r-1)}{\rho^{(1-r)}} \\ &= M(1-r) \end{aligned} \quad (22)$$

This is same as Eq. (17) which gives the optimal tradeoff as asymptotic high-SNR.

F. Observations

For a rate adaptive SIMO-MRC scheme, it is observed that the DMT, derived from the outage probability, at asymptotically high signal to noise ratio is the same as optimal DMT. This is observed from Eq. (16) and Eq. (17). Hence the rate adaptive SIMO-MRC scheme is asymptotically optimal in the DMT framework. Since Eq. (22) is also same as Eq. (17), the DMT at finite signal to noise ratios also leads to the optimal DMT in the asymptotic sense. From Fig. 3, it is seen that for a rate adaptive SIMO-MRC scheme, the diversity gain at a fixed r and fixed SNR ρ is always less than the optimal diversity gain.

The DMT analysis has a useful application. The finite-SNR diversity gain provides an estimate of the additional power required to decrease the outage probability by a target amount. This diversity characterization is useful in determining suitable rate adaptation strategies at realistic SNRs [3].

As mentioned in Section I.C, [6] provides the performance of SIMO-MRC in terms of BER or outage probability. But the diversity performance of SIMO-MRC provided in [6] cannot be directly compared with the diversity performance provided by the DMT in Fig. 3. This is due to the fact that the previous analysis of SIMO-MRC in [5], [6] are for fixed rate systems whereas the DMT analysis performed on SIMO-MRC gives the performance of the SIMO-MRC for a rate-adaptive system.

IV. Conclusion

Under the assumption of a rate-adaptive system, the closed form expression for DMT of SIMO-MRC at finite signal to noise ratios has been derived. Upon comparison of the diversity gain at a fixed and finite signal to noise ratio with the optimal diversity gain in the asymptotic high-SNR case indicates that even for a signal to noise ratio of around 30 dB, the rate adaptive SIMO system cannot achieve the optimal diversity gain. Future work includes the analysis of the DMT of other rate-adaptive SIMO diversity combining schemes at finite signal to noise ratios.

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