



# Trellis Coded Block Codes and Applications

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**Abstract**—This paper presents a Trellis Coded Modulation (TCM) scheme that is built from Linear Block Codes (LBC). These codes are referred to as Trellis Coded Block Codes (TCBC), and unlike conventional TCM, can be used for both discrete as well as continuous channels. This has been made possible by utilizing a new algebraic structure discovered by the authors. This structure allows one to partition any linear block code into sub-codes with a constant distance between the code words. As in the conventional TCM scheme, the proposed code-set partitioning is used to increase the minimum distance between code words. Another feature of TCBC is that implementations with and without bandwidth expansion are possible. Simulation results are provided to demonstrate the coding gain obtainable in binary symmetric channel (BSC) and additive white Gaussian noise (AWGN) channel.

## I. INTRODUCTION

Ungerboeck introduced Trellis coded modulation (TCM) [1], [2], where it was shown that coding gain can be obtained by partitioning the signal constellation to improve the minimum distance between constellation points, along with a trellis code to select the sub-set. The trellis structure introduces memory in the sequence of symbols transmitted, which results in a bandwidth expansion unless a higher order constellation is used. When used with constellation expansion, TCM improves the minimum distance between code (or constellation) symbols at the same Signal to Noise Ratio (SNR), thereby improving the Bit Error Rate (BER) performance. TCM was extended to multi-dimensional codes by Wei [3], which utilizes the dense sub-lattices in higher dimensional constellations and also provides (additional) shaping gain apart from the coding gain obtained from the TCM. Forney generalized TCM codes as coset codes using the lattice partitioning argument [4], [5].

TCM encoder directly maps the encoded bits into modulation symbols and hence they are suitable for continuous channels like AWGN channel. Several authors have created efficient codes by mapping the (finite group) linear block codes into Euclidean space codes. In these mappings, a proportionality between the Hamming and Euclidean distances is maintained. Combining block codes and modulation is not new, and is known as block coded modulation in the literature [17]. However, if one needs to apply the TCM for discrete channels, there is no known method of code partitioning to maximize the Hamming distance.

Imai and Hirakawa [6] introduced the Multi-Level Coding (MLC) technique, where multiple block codes of various

rates are mapped to suitable partitions of the constellation to enhance the error performance. Here, the constellation is binary partitioned, with one block code associated with each partition. The rates of the constituent codes are chosen carefully to maximize the Euclidean distance of the composite MLC code. Cusack [8] constructed coded modulation schemes by applying set-partitioning and Reed-Muller codes to QAM signal constellations. Sayegh [9] extended this work by using short block codes followed by a mapping procedure for PSK/QAM symbols. A search procedure was used to find the mapping that maximizes the minimum Euclidean distance based on the rate of the codes and constellation type (PSK, QAM). It was shown that an MLC built with short block codes can provide up to 6 dB coding gain with lesser complexity than the TCM.

Tanner [10] formally related the minimum Hamming distance of the component codes to the minimum Euclidean distance of the multilevel scheme, and designed powerful codes which can be decoded with an efficient recursive procedure. Ginzburg [11] used a hierarchical partition of the constellation set (group theoretically) to design efficient codes by mapping one block code for each level of partition. Kschischang et al [12] introduced block coset codes by mapping the blocks codes onto M-PSK constellations. In their work, a group property of the basic constellation that is available in their  $n$ -tuple extensions (Cartesian product of the basic set) is used to perform the mapping. Calderbank and Sloane [13] partitioned the  $N$ -dimensional Euclidean space into various cosets of a sub-lattice and generated TCM using these sub-lattices. Forney [4], [5] observed that every coded modulation scheme can be described in an algebraic frame work of groups and group partitions. Pollara et al [14] built codes using finite state machines (FSM) and assigned cosets from block/convolutional codes to various transitions in the FSM, in order to improve the minimum distance of the overall code.

In all of the past works, to the best of the authors' knowledge, there is no set partitioning of block codes in the Hamming space, except for coset decomposition and association scheme<sup>1</sup> which could help in generating codes with large minimum distances. If block codes with large minimum distances can be derived from known block codes,

<sup>1</sup>Note that the coset decomposition-based partitioning need not result in cosets with constant distance among elements in the coset. And an association scheme groups code words into code-pairs at several distances.



then one can use any standard mapping scheme to generate the Euclidean space codes with such large minimum distance. This paper addresses some of the issues mentioned above. The main contributions in this paper are:

- A new result describing partitioning of linear block codes into constant distance sub-codes.
- An encoder/decoder structure which utilizes the constant distance sub-code partitioning in Hamming space and hence is usable in both discrete and continuous channels is proposed. The codes constructed using such encoder structure are referred to as Trellis Coded Block Codes (TCBC).
- Expressions for the BER of the TCBC are derived.

This paper is organized as follows. Section II provides a brief review of trellis coded modulation and states the result on the algebraic structure of linear block codes. Section III describes the construction of TCBC using code partitioning. Section III-A describes the structure of the encoder and decoder for TCBC with and without bandwidth expansion. Section IV presents the bit error bounds for TCBC. Section V provides simulation results to illustrate the performance improvements that can be obtained using the proposed code partitioning. Section VI concludes the paper.

## II. CODE PARTITIONING AND TCM

There are four known ways of partitioning linear block codes: (i) the sub-codes obtained by a coset decomposition [4], (ii) sub-codes of smaller length which constitute the mother code by concatenation [7], (iii) an association scheme to group code-pairs with a given Hamming distance between them [16], and (iv) decomposing  $N$ -tuple codes into cosets using finite groups that exploit the algebraic structure available in Cartesian product space [12]. Here we describe another algebraic structure which allows one to partition code words into disjoint sub-code sets.

### A. Constant Distance Sub-code Partitioning

*Theorem 1:* Any linear block code can be partitioned into the union of disjoint sub-codes with equal distance between codewords (in the sub-codes). That is

$$C = \bigcup_{i=1}^L C_i \quad (1)$$

such that  $C_i \cap C_j = \{\phi\}$ ,  $1 \leq i, j \leq L$ ,  $i \neq j$ ; where  $C_i$  are constant distance sub-codes (not necessarily linear sub-codes),  $0 < L < 2^k$  is the number of constituent sub-codes and  $\{\phi\}$  is a null set.

*Proof:* Omitted due to lack of space.

Note that that  $L$  depends on the number of sub-codes one wishes to obtain. That is, for an  $(n, k, d)$  code, the maximum value of  $L$  is  $2^k$  when the code is trivially partitioned into sub-codes with single elements.  $L$  is  $2^{k-1}$  when the code is trivially partitioned into sub-codes with pairs of elements. A non-trivial constant distance sub-code must contain at least 3 code-words in each sub-code, such that distance between

any two code-words within any sub-code is constant. Thus, if the number of code-words in each sub-code is increased by adding more elements without compromising on the constant distance property, one obtains a unique partition with *maximal constant distance sub-codes*. The sub-codes are not necessarily linear since cosets of any constant distance sub-code are also constant-distance sub-codes. But, one has to choose the coset-leads such that the resulting cosets are disjoint.

**Examples:** The following simple examples illustrate the use of Theorem 1. The Hamming (7,4) code can be partitioned into 2 sub-codes<sup>2</sup> as follows:  $C_1 = \{0, 1, 6, 7, 10, 11, 12, 13\}$  and  $C_2 = \{2, 3, 4, 5, 8, 9, 14, 15\}$ . All the code words in each sub-set are at equal Hamming distance (equal to 4) from each other. Another example is maximum length shift register (MLSR)<sup>3</sup> (6,3) code which can be partitioned into  $C_1 = \{1, 2, 5, 6\}$ ,  $C_2 = \{0, 3, 4, 7\}$ . The Hamming distance between any pair of elements (for  $i^{th}$  sub-code, it is denoted as  $d_H(C_i)$ ) in both sub-codes  $C_1$  and  $C_2$  is 4; although  $D_{min}$  is 3. Similarly, the Reed-Solomon code RS (3,2,2) [16] can be partitioned into  $C_1 = \{0, 1, 2, 3\}$ ,  $C_2 = \{4, 5, 6\}$ ,  $C_3 = \{7, 8, 9\}$ ,  $C_4 = \{10, 11, 12\}$  and  $C_5 = \{13, 14, 15\}$ . Here, the sub-codes  $C_1, C_2, C_3$  have  $d_H(C) = 2$  and  $C_4, C_5$  have  $d_H(C) = 3$ .

### B. Trellis Coded Modulation

Since the proposed TCB uses ideas from TCM, it is briefly described here. TCM[2] uses set partitioning of the modulation symbol constellations so that the minimum Euclidean distance between trellis codes generated, is maximized. Since multiple levels of set-partitioning can be done with increasing minimum distance between symbols, TCM can utilize any level of partitioning as per the minimum distance requirements. However there is a limit on the maximum separation between the finite set of constellation symbols, which limits the maximum gain that can be obtained by using a given constellation. Thus, a TCM code tries to maximize the minimum distance between coded symbols (constellation points) by choosing the sub-sets carefully.

## III. TRELIS CODED BLOCK CODES

It was shown in [4] that one can obtain a coding gain from coset codes, by using the coset (sub-code) partition structure and selecting the cosets using a trellis code output. In [14], the minimum distance was improved by assigning a code (or a coset) to the transition of fully connected finite state machines. In the proposed TCBC, improvement in the minimum distance is achieved through the constant distance sub-code partitioning described in Theorem 1. Thus, a new family of codes based on sub-code partitioning is introduced which are based on existing linear block codes. Improved error performance is obtained since minimum distance of TCBC is increased by combining a trellis code and sub-code partitioning using Theorem 1.

<sup>2</sup>The codewords in the sub-code are denoted by indices which are decimal equivalent of binary data vectors in the code  $C$ . The binary data vectors are binary  $k$ -tuples which get multiplied by the generator matrix  $G$  to generate the codewords in  $C$ .

<sup>3</sup>The codewords in MLSR (6,3) are given in octal notation:  $C = \{00,16,23,35,47,51,64,72\}$ .

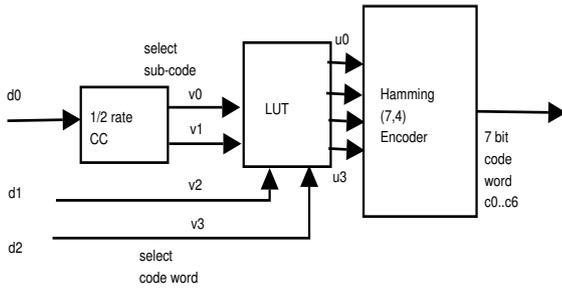


Fig. 1. Encoder for TCBC using sub-codes of Hamming (7,4) code.

For example, consider the TCB  $(n, 3)$  code with 3 input bits and  $n$ -output bits. Let one of the input bits be fed into a  $\frac{1}{2}$  rate convolutional code whose output selects one of 4 sub-codes (partitions) and the 2 remaining input bits select one of the code-words in the selected sub-code, thus selecting an  $(n, k = 3)$  TCB code. The minimum distance of this hybrid code is defined by the smaller of the (constant) minimum distance between code words in each sub-code and the  $d_{free}$  of the trellis (convolutional) code. Here  $n$  is the length of the block code  $\mathcal{C}$  with  $2^{k+1}$  codewords which is partitioned into 4 sub-codes with  $2^{k-1}$  elements in each. If  $d_{free}$  of the trellis code is chosen to be large enough, the minimum Hamming distance of TCBC is determined by the distance between the code-words in the sub-code (assigned to parallel transitions of the trellis code). Thus by choosing the sub-code partitioning appropriately, one can obtain a coding gain similar to TCM/coset codes.

### A. Encoder and Decoder for TCBC

Fig. 1 shows an example of a TCB encoder that uses the Hamming (7,4) code as the underlying binary block code. The encoder comprises a trellis code, which selects the sub-code indices  $(v_0, v_1)$ , a look-up-table (LUT) which selects the data word based on sub-code indices and the remaining input data bits  $(v_2, v_3)$ , followed by the block code encoder. The entries of the LUT enforces the sub-code partition structure. For example, for the 4 sub-code partitioning of the Hamming (7,4) code as  $\mathcal{C}_1 = \{0, 1, 6, 7\}$ ,  $\mathcal{C}_2 = \{10, 11, 12, 13\}$ ,  $\mathcal{C}_3 = \{2, 3, 4, 5\}$  and  $\mathcal{C}_4 = \{8, 9, 14, 15\}$ , if  $(v_0 v_1 v_2 v_3 = 0000)$  the LUT will select 1<sup>st</sup> codeword in  $\mathcal{C}_1$ , if  $(v_0 v_1 v_2 v_3 = 1111)$  the LUT will select 4<sup>th</sup> codeword in  $\mathcal{C}_4$  and so on. As an example, complete LUT for the TCBC (7,3) using the Hamming (7,4) code is given in Table I. Once data-words to be encoded are determined, the output of the LUT, denoted  $(u_0, u_1, u_2, u_3)$ , is sent to a conventional Hamming (7,4) block encoder which generates the 7-bit codeword. The coded bits can be transmitted on BSC as is, or mapped into BPSK symbols and transmitted on AWGN channels. However, if one does not want to expand the bandwidth needed<sup>4</sup> in the AWGN channel, one can adopt the MLC scheme described below.

<sup>4</sup>When one encodes raw data bits, the number of output bits are more than that of the input bits and hence one needs to send more bits in the same amount of time. This is referred to as bandwidth expansion in the coding theory literature.

$v_0 v_1 v_2 v_3$	$u_0 u_1 u_2 u_3$	$v_0 v_1 v_2 v_3$	$u_0 u_1 u_2 u_3$
0000	0000	1000	1100
0001	0001	1001	1101
0010	0110	1010	1010
0011	0111	1011	1011
0100	0100	1100	1000
0101	0101	1101	1001
0110	0010	1110	1110
0111	0011	1111	1111

TABLE I  
LUT MAPPING FOR TCB (7,3) CODE.

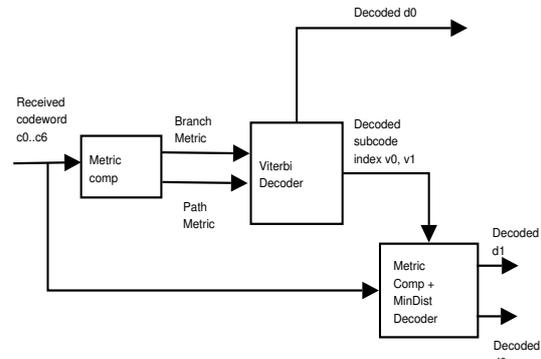


Fig. 2. Decoder for TCBC using sub-codes of Hamming (7,4) code.

The decoder structure is shown in Fig. 2. A trellis decoder (Viterbi decoder, Fano decoder etc.) is used for detecting the sub-code index. Once sub-code indices are estimated (after tracing data bits back as in Viterbi decoder), this information is used to compute the distance (Hamming/Euclidean) between the transmitted (block) code word and the codewords in the selected sub-code/coset. The codeword with the least distance is used to estimate the remaining data bits. In this paper, three models for transmission of the coded bits are considered.

1) *BSC with bandwidth expansion*: If the bandwidth expansion is tolerable, the coded bits are directly transmitted on a BSC.

2) *AWGN With bandwidth expansion*: The coded bits can be sent on the AWGN channel by simply mapping the binary symbols onto BPSK symbols with amplitude  $\sqrt{\mathcal{E}_s}$ . Thus, the  $\mathcal{D}_{min}^2$  for the AWGN channel is  $\mathcal{E}_s d_H(\mathcal{C})$ . The decoder uses Euclidean distance as the metric instead of Hamming distance. For example, a rate  $\frac{1}{3}$  code (0.33 bits/symbol) can be obtained by encoding using TCB (6,2) code followed by BPSK mapping.

3) *AWGN without bandwidth expansion*: When bandwidth expansion is not allowed, MLC encoding schemes come in handy to realize the encoder and multi-stage decoder (MSD). For example, a rate 1 (1 bit/symbol) encoder can be obtained by combining the outputs of TCB (7,3) code and Hamming (7,4) code and QPSK symbol mapping. That is, transmit 7 QPSK symbols for each codeword pair output from the 2 block codes. The decoder decodes the TCB(7,3) code first and the decoder output is used to rotate the QPSK symbols before computing the metric to estimate the output of the Hamming (7,4) encoder. This estimated Hamming (7,4) code is decoded by a regular Hamming (7,4) decoder to recover data bits. Figs. 3 and 4 describe the encoding and decoding steps using

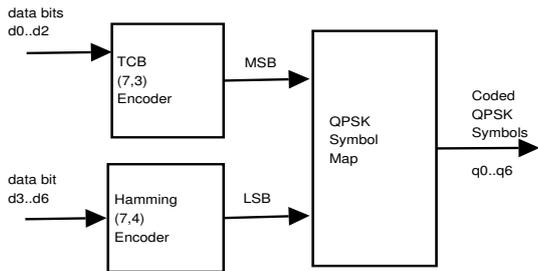


Fig. 3. Encoder for MLC comprising TCB(7,3) and Hamming (7,4) code.

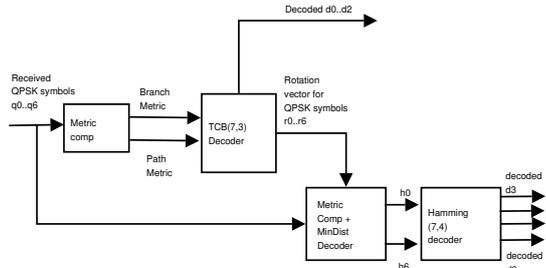


Fig. 4. Decoder for MLC comprising TCB(7,3) and Hamming (7,4) code.

the MLC scheme and the MSD.

#### IV. COMPUTATION OF BER

BER bounds for the TCBC can be computed using the theory developed for MLC since TCBC is also an MLC with 2 levels. For a 2 level code, the BER can be written as [15]

$$P_e = \frac{R_1 P_{e,1} + R_2 P_{e,2}}{R} \quad (2)$$

where  $P_{e,1}$  and  $P_{e,2}$  are the BERs of the level 1 code  $(n, k_1)$  and level 2 code  $(n, k_2)$  respectively. The  $P_{e,1}$  can be computed since it is a trellis code. The BER expressions can be simplified by assuming that the BER is dominated by the codes at distance  $d_{free}$  [17]. In the case of TCBC, the level 1 code is the CC/trellis code and the level 2 code is the sub-code of the LBC. It can be shown that

$$P_e^{BSC} \approx \frac{B_{d_{free}} R_1}{R} 2^{d_{free}} [p(1-p)]^{\frac{d_{free}}{2}} + \frac{\binom{n}{\tilde{d}} R_2}{R} p^{\tilde{d}} (1-p)^{n-\tilde{d}} \quad (3)$$

$$P_e^{AWGN} \approx \frac{B_{d_{free}} R_1}{R} Q\left(\sqrt{\frac{2d_{free} R_1 \mathcal{E}_b}{\sigma_n^2}}\right) + \frac{R_2}{R} Q\left(\sqrt{\frac{2\tilde{d} R_2 \mathcal{E}_b}{\sigma_n^2}}\right) \quad (4)$$

where  $B_{d_{free}}$  is the number of neighbours at distance  $d_{free}$ ,  $R_1$  is the rate of the level 1 code,  $R_2$  is the rate of level 2 code and  $R$  is overall rate of the MLC code. Also,  $\tilde{d} = \min_i d_H(C_i)$ . Detailed derivation of the above equations are omitted due to lack of space.

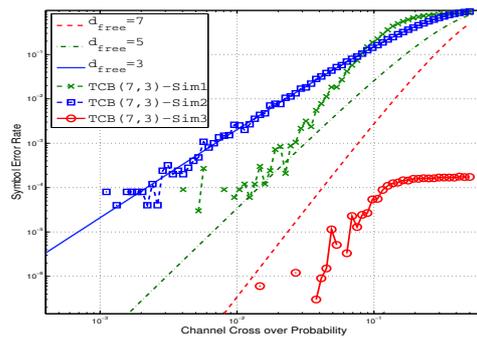


Fig. 5. SER performance of TCBC (7,3) using Hamming (7,4) code in BSC.

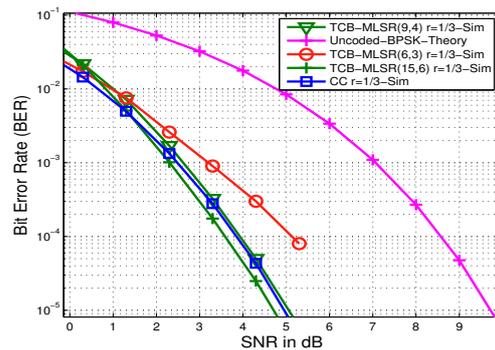


Fig. 6. Performance comparison of CC and TCBC using MLSR(6,3), MLSR(9,4), MLSR(15,6) in AWGN Channel.

#### V. SIMULATION RESULTS

To demonstrate the coding gain in a BSC, Monte-Carlo simulations were performed with TCBC based on the Hamming (7,4) code. The Hamming (7,4) code is partitioned into 4 sub-codes (with  $d_H(C_i) = 4$ ) for  $d_{min} = 3$  case (solid line in Fig. 5) and partitioned into 8 sub-codes (with  $d_H(C_i) = 7$ ) for  $d_{min} = 5$  and 7 (dash-dotted line and dashed line in Fig. 5). The Symbol Error Rate (SER) was computed using  $10^5$  data bits (except for the  $d_{min} = 7$  case where  $10^6$  bits were used). The theoretical performance for various  $d_{min}$  distances are plotted for comparison. To show the higher coding gain of TCBC,  $\frac{2}{3}$  trellis code with  $d_{free} = 5$  is used for  $d_{min} = 3, 5$  cases and a trellis code with  $d_{free} = 7$  is used for  $d_{min} = 7$  cases. It can be observed that the slope of the SER curve matches that of theoretical curve (of a hypothetical code of the same length and minimum distance) more closely at low channel cross over probability (equivalently high SNR).

To demonstrate the coding gain in AWGN channel with bandwidth expansion, TCB codewords mapped to BPSK are sent over the AWGN channel. Fig. 6 compares the BER performance of the rate  $\frac{1}{3}$  TCBC using MLSR codes (for different code lengths 6, 9 and 15) with convolutional code of similar coding gain. The sub-codes of MLSR (6,3) code were selected by the rate  $\frac{1}{2}$  convolutional code (defined by the polynomials (7;5) whose  $d_{free}$  is 5) and the sub-codes of MLSR (9,4) code<sup>5</sup> were selected by the rate  $\frac{2}{3}$  trellis code

<sup>5</sup>The MLSR(9,4) code words in octal notation are  $\mathcal{C} = \{000, 075, 107, 172, 217, 262, 310, 365, 436, 443, 531, 544, 621, 654, 726, 753\}$ .

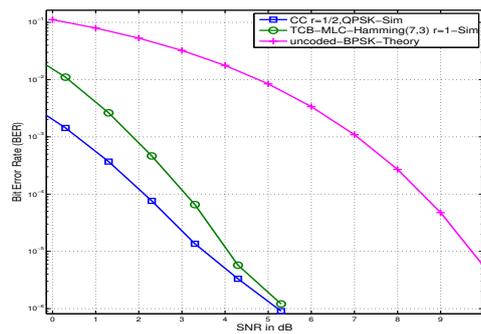


Fig. 7. BER performance of Rate=1 bit/symbol TCBC using Hamming (7,4) in AWGN channel.

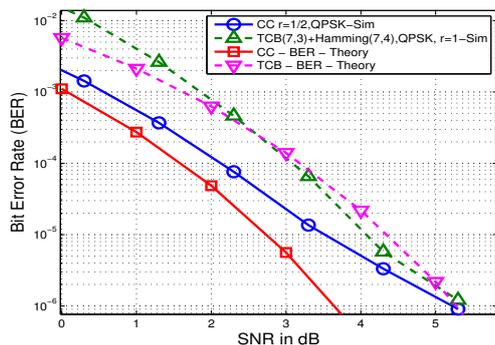


Fig. 8. Error bounds Vs simulation values for Rate=1 bit/symbol TCBC in AWGN channel.

(defined by the polynomials (7 4 1;2 5 7) whose  $d_{free}$  is 5). They are compared with the performance of rate  $\frac{1}{3}$  CC code (defined by the polynomials (17 15 13) whose  $d_{free}$  is 10) It can be seen that MLSR(15,6) code outperforms the CC code. Both convolutional codes has constraint length  $K = 3$ . Note that the comparison is done here with the performance of a convolutional code of same rate and trellis complexity. It can be seen that TCBC performs as well as or better than convolutional code at high SNRs. At low SNRs, performance is worse than the convolutional code due to sub-optimality of the MSD and error propagation between the levels of the decoder.

Fig. 7 compares the performance (for rate=1 bit/symbol) of TCBC(7,3) based MLC (along with the Hamming (7,4) code as shown in Fig. 3) with a rate  $\frac{1}{2}$  convolutional code whose output is mapped onto QPSK symbols. It can be seen that the performance is similar at high SNR. Fig. 8 demonstrates the theoretical BER bounds (4) for the convolutional code as well as the TCB code. It can be seen that the simulation result matches the theoretical curve fairly closely.

### VI. CONCLUSIONS

In this paper, a new class of codes referred to as trellis coded block codes was introduced, which can be used in discrete as well as continuous channels. This framework can be used to obtain a coding gain starting from any LBC. At a small loss ( $\frac{1}{n}$ ) in the rate, the BER performance can be improved multi-fold. This is made possible via the constant distance sub-code

partitioning of block codes shown in this paper. A simple encoder/decoder structure (which uses the block encoder as is) is given, and the decoder uses multi-stage decoding used to decode multi-level codes (MLC). The structure of the encoder/decoder with or without bandwidth expansion was devised for both discrete as well as continuous channels. Expressions for BER of the TCB code was obtained based on the theory of MLC decoding. Simulation results demonstrated the coding gain in BSC and AWGN channels. One can treat TCBC as a concatenated code and apply iterative decoding between multi-levels to minimize the error propagation effects at low SNRs, which would be pursued as an extension of this work.

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