A Class of Time-Frequency product optimized Biorthogonal Wavelet Filter Banks

Ritesh Kolte, Pushkar Patwardhan, Vikram Gadre
Department of Electrical Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076.
Email: ritesh@ee.iitb.ac.in, pushkarp@tensilica.com, vmgadre@ee.iitb.ac.in

***Abstract*—The time-frequency product of any function in $L^2(\mathbb{R})$ is bounded by the uncertainty principle. This paper presents a method to design linear phase biorthogonal filter banks with the time-frequency localization as the optimality criterion. The design philosophy is to optimize the time-frequency product of the iterated wavelet, after fixing the number of vanishing moments of the analysis and synthesis lowpass filters, by adjusting a single parameter.**

***Index Terms*—biorthogonal filter banks, linear phase, wavelets, time-frequency localization.**

I. Introduction

The analysis of non-stationary signals involves a trade-off in the sense that one has to sacrifice time localization in order to achieve better frequency localization and vice versa. If we assume that the function is normalized, the uncertainty principle states that the product of the time-spread (defined in the variance-sense) and frequency spread is bounded below by 0.25. The design of filter banks has generally been developed keeping in mind criteria such as energy compaction, non-aliasing energy ratio, coding gain etc. However time-frequency localization is a fundamental limitation in signal processing in one dimension as well as in higher dimensions, and literature on time-frequency localization optimized filter bank design has been relatively sparse. In [1], the authors have attempted to optimize the time-frequency product of the continuous time orthonormal wavelet by expressing it in terms of the corresponding discrete filter coefficients. In [2], the authors designed time-frequency localized orthonormal wavelets by optimizing the lattice structure parameters of the corresponding paraunitary filter bank. However, as is well known, linear phase cannot be achieved by orthogonal filters except in the 2-tap case. Hence, biorthogonal wavelets, which have been extensively used especially in image coding, cannot be designed using this method. B-spline biorthogonal wavelets have been analyzed for their time-frequency resolution property in [3]; however [3] does not give a design strategy to obtain time-frequency resolution optimized wavelets.

In this paper, we give a method to design biorthogonal wavelets optimized for their time-frequency localization. The basic methodology employed is as follows: By using the single-parameter parametrization suggested in [4], we construct regular biorthogonal filter banks. Then we optimize the parameter to obtain that filter bank which corresponds to the analysis wavelet with the minimum time-frequency product.

The rest of the paper is organized as follows: In section II, we study in brief the theoretical background and previous work relevant to our problem. In section III, the method used to construct minimum time-frequency product wavelets for a given set of lengths of the analysis and synthesis lowpass filters is described in detail. In section IV, the numerical results are presented and interpreted. Finally, section V gives the concluding remarks and directions for further research.

II. Background

A. Perfect Reconstruction Filter Banks

A general one-dimensional 2-channel perfect reconstruction (PR) filter bank, or a biorthogonal filter bank, is shown in Fig. 1.

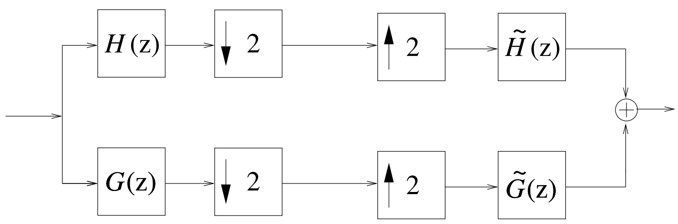

Figure 1. A general one-dimensional 2 channel filter bank

The conditions for PR are well known [5]. We shall be dealing with FIR filters throughout this paper. Associated with this (regular) biorthogonal filter bank is a biorthogonal wavelet system consisting of two dual scaling functions $\phi(x)$ and $\tilde{\phi}(x)$, and the corresponding wavelets $\psi(x)$ and $\tilde{\psi}(x)$. These 4 functions satisfy the following two-scale equations:

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k) \tag{1}$$

$$\tilde{\phi}(x) = \sqrt{2}\sum_k \tilde{h}_k \tilde{\phi}(2x-k) \tag{2}$$

$$\psi(x) = \sqrt{2}\sum_k g_k \phi(2x-k) \tag{3}$$

$$\tilde{\psi}(x) = \sqrt{2}\sum_k \tilde{g}_k \tilde{\phi}(2x-k) \tag{4}$$

where h_k, $\tilde{h}_k$, g_k and $\tilde{g}_k$ are finite length real-valued sequences, corresponding to the 2 lowpass filters and the 2 highpass filters in the biorthogonal wavelet system respectively. First, we define

$$H(\xi) := \frac{1}{\sqrt{2}} \sum_k h_k e^{-jk\xi},$$

$$\tilde{H}(\xi) := \frac{1}{\sqrt{2}} \sum_k \tilde{h}_k e^{-jk\xi},$$

$$G(\xi) := \frac{1}{\sqrt{2}} \sum_k g_k e^{-jk\xi},$$

$$\tilde{G}(\xi) := \frac{1}{\sqrt{2}} \sum_k \tilde{g}_k e^{-jk\xi}.$$

Then we choose

$$G(\xi) = e^{-j\xi} \overline{\tilde{H}(\xi + \pi)}$$

and

$$\tilde{G}(\xi) = e^{-j\xi} \overline{H(\xi + \pi)}$$

so that the perfect reconstruction condition becomes

$$H(\xi) \overline{\tilde{H}(\xi)} + H(\xi + \pi) \overline{\tilde{H}(\xi + \pi)} = 1. \tag{5}$$

($\overline{H(\xi)}$ denotes the complex conjugated filter of $H(\xi)$). A necessary condition for the biorthogonal filter bank to correspond to a regular biorthogonal wavelet system is: $H(0) = \tilde{H}(0) = 1$ and $H(\pi) = \tilde{H}(\pi) = 0$.

Given $H(\xi)$ and $\tilde{H}(\xi)$, approximations to $\phi(x)$, $\tilde{\phi}(x)$, $\psi(x)$ and $\tilde{\psi}(x)$ are generated by the cascade algorithm [5].

B. Parametrization of linear phase biorthogonal filter banks

The authors in [4] present the following parametrization of the analysis lowpass filter. Here we shall restrict ourselves to filters with an odd number of taps. Then, the analysis and synthesis lowpass filters are restricted to have an even number of vanishing moments (Definition 2 in [4]), say $2l$ and $2\tilde{l}$. Then if $H(\xi)$ has the form

$$H(\xi) = \cos^{2l}\left(\frac{\xi}{2}\right)\left[a + (1-a)\cos\xi\right], \tag{6}$$

the unique lowest length synthesis lowpass filter such that (5) is satisfied is given by

$$\tilde{H}(\xi) = \cos^{2\tilde{l}}\left(\frac{\xi}{2}\right) Q\left(\sin^2\left(\frac{\xi}{2}\right)\right), \tag{7}$$

where $Q(x)$ is a polynomial of degree $l + \tilde{l}$, and whose coefficients have a closed-form expression depending on the parameter a and $a \notin \{0, 1\}$. Note that these definitions imply that along with symmetry, the filters also satisfy the necessary conditions for regularity: $H(0) = \tilde{H}(0) = 1$ and $H(\pi) = \tilde{H}(\pi) = 0$ if $l, \tilde{l} \geq 1$. By this parametrization, the lengths of the filters $H(\xi)$ and $\tilde{H}(\xi)$ are $2l+3$ and $2l+4\tilde{l}+1$.

For details, we refer the reader to section 3 in [4].

C. A sufficient condition for regularity

We shall make use of the following result [6]:

Theorem 1: If

$$H(z) = \left(\frac{1 + z^{-1}}{2}\right)^N P(z),$$

then $\phi(x)$ is a continuous function if

$$\sup_{\omega \epsilon [0, 2\pi]} \left\{\left|P\left(e^{j\omega}\right)\right|\right\} < 2^{N-1} \tag{8}$$

An important point to note is that unlike paraunitary filter banks, in the case of biorthogonal filter banks, we have to check the regularity on both the analysis and the synthesis sides of the filter bank.

D. Defining Time-Frequency Localization

Suppose $f(x)$ is a function in $L^2(\mathbb{R})$ with unit norm. The Fourier Transform of $f(x)$ is given by

$$F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-j\xi x} dx.$$

Without loss of generality, we can assume that the function is centred at the origin in time, since the variance of the function is invariant under translation.

Define the time-spread of the function as:

$$\sigma_x^2 := \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx.$$

Since $\psi(x)$ and $\tilde{\psi}(x)$ are bandpass functions, we define the frequency spread as follows [2]:

$$\sigma_\xi^2 := \frac{1}{\pi} \int_0^{\infty} (\xi - \xi_0)^2 |F(\xi)|^2 d\xi,$$

where

$$\xi_0 = \frac{1}{\pi} \int_0^{\infty} \xi |F(\xi)|^2 d\xi.$$

The Uncertainty Principle states that if $f(x)$ is a function in L^2 such that $\sqrt{|x|} f(x) \to 0$ as $|x| \to \infty$, the product of the time-spread and the frequency spread, known as the tf-product, cannot be less than 0.25, i.e.

$$\sigma_x^2 \sigma_\xi^2 \geqq 0.25 \tag{9}$$

III. Methodology

To construct biorthogonal filter banks, we first fix the number of vanishing moments in $H(z)$ and $\tilde{H}(z)$, say l and $\tilde{l}$. By using the parametrization proposed in [4], we construct linear phase biorthogonal filter banks and then the corresponding continuous time wavelets $\psi(x)$ and $\tilde{\psi}(x)$. Note that since the forms of $H(z)$ and $\tilde{H}(z)$ are different, the pair of wavelets obtained by interchanging the analysis and synthesis filters for a given $\left\{l, \tilde{l}\right\}$ will not be the same. Hence, in one approach, we minimize the tf-product of the synthesis wavelet $\tilde{\psi}(x)$ and in the second approach, we minimize the tf-product of the analysis wavelet $\psi(x)$. Since this is a biorthogonal system, once we obtain the pair $\left\{\psi(x), \tilde{\psi}(x)\right\}$ from any one of the approaches, we can reverse the order and use $\tilde{\psi}(x)$ as the analysis wavelet and $\psi(x)$ as the synthesis wavelet. Usually, the smoother of the two is used at the synthesis side to get better compression and robustness to error. However, in this paper, we shall be concerned only with obtaining the minimum possible tf-product for any of the two wavelets from the two approaches and use that wavelet as the analysis wavelet.

Since an exact expression of the tf-product of either of the two wavelets is not known in terms of the parameter a, we used numerical methods to arrive at the results. Unconstrained optimization failed to work in this case as the obtained system failed to be regular in many cases. To circumvent this limitation, we evaluated the tf-product for several values of a. It was found that for a wide choice of values of the pair $\left\{l, \tilde{l}\right\}$, both $\psi(x)$ and $\tilde{\psi}(x)$ turned out to be continuous (i.e. regular) only if a took values in a specific range. We evaluated the tf-product on a set of finely-spaced values in this range. From this, taking the value of a corresponding to the minimum tf-product as the initial condition, we now performed constrained optimization, with the constraint being the regularity condition in (8). Applying constrained optimization without this carefully chosen initial condition didn't result in attainment of the global minimum, as was observed empirically.

IV. Results

In Table I, we give some examples of biorthogonal filter banks designed by the above method. One limitation of this parametrization is that for certain values of the pair $\left\{l, \tilde{l}\right\}$, there is no value of a for which both $H(z)$ and $\tilde{H}(z)$ satisfy the sufficient condition (8) for regularity. A general rule appears to be to keep the value of l low compared to that of $\tilde{l}$, and then minimize the tf-product of the wavelet $\tilde{\psi}(x)$ (If we try to minimize the tf-product of $\psi(x)$, we don't obtain the lowest possible value for the given $\left\{l, \tilde{l}\right\}$). The $\tilde{\psi}(x)$ thus obtained may be used as the analysis wavelet, rendering $\psi(x)$ as the synthesis wavelet. E.g., in the case $\left\{l, \tilde{l}\right\} = \{2, 7\}$ (i.e. the lengths of $H(z)$ and $\tilde{H}(z)$ are 7 and 33 respectively), if we minimize the tf-product of $\psi(x)$, the tf-products of $\psi(x)$ and $\tilde{\psi}(x)$ are found to be 0.6148 and 0.2632 respectively, while if we minimize the tf-product of $\tilde{\psi}(x)$, we get the tf-products of the two wavelets to be respectively, 0.6989 and 0.2570. Hence we use the results from the latter case and use $\tilde{\psi}(x)$ as our analysis wavelet. See Fig. 3 and Fig. 4.

To enhance the reproducibility of this research, we shall go through one example in detail. Consider the case: $\left\{l, \tilde{l}\right\} = \{1, 1\}$, i.e. the lengths of the filters are 5 and 7. For this, the coefficients of the two filters are obtained from [4] as: $\left(\frac{1-a}{8}, \frac{1}{4}, \frac{1+a}{4}, \frac{1}{4}, \frac{1-a}{8}\right)$ and $\left(\frac{a^2-3a+2}{16a}, \frac{a-2}{8a}, \frac{7a-a^2-2}{16a}, \frac{a+2}{4a}, \frac{7a-a^2-2}{16a}, \frac{a-2}{8a}, \frac{a^2-3a+2}{16a}\right)$.

From (6) and the sufficient condition (8), we can see that the sufficient condition for regularity of $H(z)$ becomes

$$\sup\left\{|a + (1-a)\cos\xi|\right\} < 2$$

i.e.

$$\sup\left\{1, |2a-1|\right\} < 2, \tag{10}$$

which implies $\frac{-1}{2} < a < \frac{3}{2}$. From (7), the sufficient condition for the filter $\tilde{H}(z)$ becomes

$$\sup\left\{\left|Q\left(\sin^2\left(\frac{\xi}{2}\right)\right)\right|\right\} < 2,$$

which in this case becomes

$$\sup\left\{1, \left|2a - 7 + \frac{8}{a}\right|, \left|\frac{8 - 16a + 8a^2 - a^3}{4(a-2)(a-1)}\right|\right\} < 2. \tag{11}$$

An excerpt of the plot of the LHS in (11) vs. a is shown in Fig. 2. A horizontal line with y-axis intercept of 2 is also drawn to illustrate the range of values a for which the sufficient condition (11) is satisfied. We take the intersection of the values of a satisfying (10) and (11) and in this set, construct a grid of values spaced by a sufficiently small value, say 0.2. Then for every a, we construct the biorthogonal filter bank and evaluate the tf-products of the corresponding wavelets $\psi(x)$ and $\tilde{\psi}(x)$. If we are interested in minimizing the tf-product of $\psi(x)$, we observe that among the selected values of a, the tf-product of $\psi(x)$ attains a minimum value of 3.5484 at $a = 1.3$. Using this value of a as the initial condition in the `fmincon` function in MATLAB along with (8) included as a non-linear constraint, we obtain the actual minimum tf-product of 2.1418 for $\psi(x)$ at $a = 1.2192$, while ensuring that the filter bank is regular. If we try to minimize the tf-product of $\tilde{\psi}(x)$, we follow a similar approach to get the minimum tf-product to be 2.4840 at $a = 1.4999$. Since we obtain a lower value in the former case, we use the results of that in Table I.

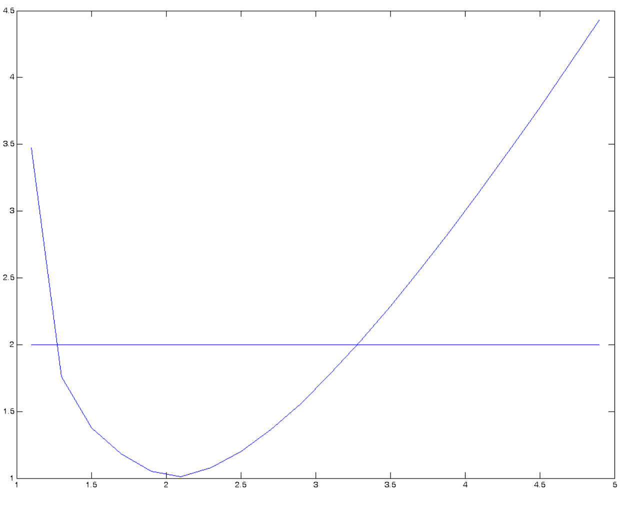

Figure 2. $\sup\left\{\left|Q\left(\sin^2\left(\frac{\xi}{2}\right)\right)\right|\right\}$ vs a for the 5/7 case

If we compare the performance of these wavelets with the orthogonal wavelets designed in [1] and [2], we see that the biorthogonal wavelets we have designed achieve a significantly lower tf-product for comparable filter lengths. E.g., the minimum tf-product obtained in [2] for the filter length 22 is 0.3239. As we can see from Table I, in the

23/5 filter bank, the minimum tf-product is 0.2621, along with reduction in length of the synthesis lowpass filter. The authors in [7] state that in the case of orthonormal wavelets, the minimum of the tf-product of the wavelet occurs when the corresponding filter has length 22. Thus, biorthogonal wavelets can achieve better performance than the best of the orthogonal wavelets in this respect.

In [3], the authors have analyzed the time-frequency resolution property of B-spline biorthogonal wavelets. It is found that the time-frequency resolution becomes poor as we increase the filter lengths. The minimum tf-product in this case turns out to be 0.4898 when the order of both $H(z)$ and $\tilde{H}(z)$ is 2.

All the above examples constrained the filter bank to be regular on both the analysis side and the synthesis side. To give an example where the regularity condition was not imposed, we performed unconstrained optimization in the 5/7 case using the `fminunc` function in MATLAB. The corresponding continuous time functions are shown in Fig. 5. $\psi(x)$ and $\tilde{\psi}(x)$ turned out to be regular and non-regular respectively. The tf-product value of $\psi(x)$ was 0.2932, comparable to the tf-product value in the constrained 21/7 case.

Table I
A FEW DESIGN EXAMPLES (ARRANGED IN DESCENDING ORDER OF THE ANALYSIS WAVELET TF-PRODUCT)

Filter lengths	analysis wavelet $\sigma_x^2\sigma_\omega^2$	synthesis wavelet $\sigma_x^2\sigma_\omega^2$	Value of the parameter a
5/7	2.1418	21.1316	1.2192
9/7	0.6554	0.4853	2.6284
23/9	0.3949	0.4776	2.9148
11/5	0.3928	17.0672	1.4999
7/9	0.3602	28.6881	2.1690
21/7	0.2902	0.5432	2.5006
17/7	0.2707	0.7469	2.9684
23/5	0.2621	22.8587	1.4875
25/7	0.2605	0.9298	3.0869
29/7	0.2584	0.6347	2.4077
33/7	0.2570	0.6989	2.5007

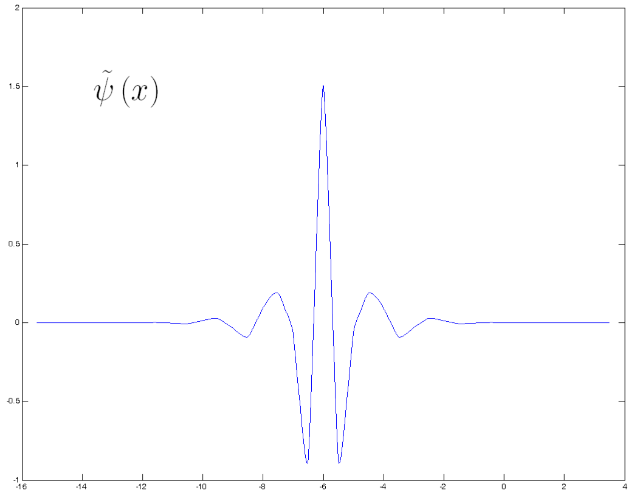

Figure 4. The synthesis wavelet for the 33/7 filter bank

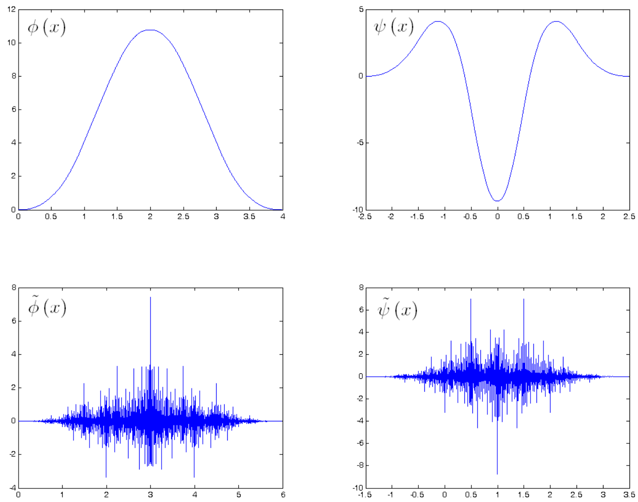

Figure 5. $\phi(x), \psi(x), \tilde{\phi}(x), \tilde{\psi}(x)$ for the unconstrained 5/7 filter bank

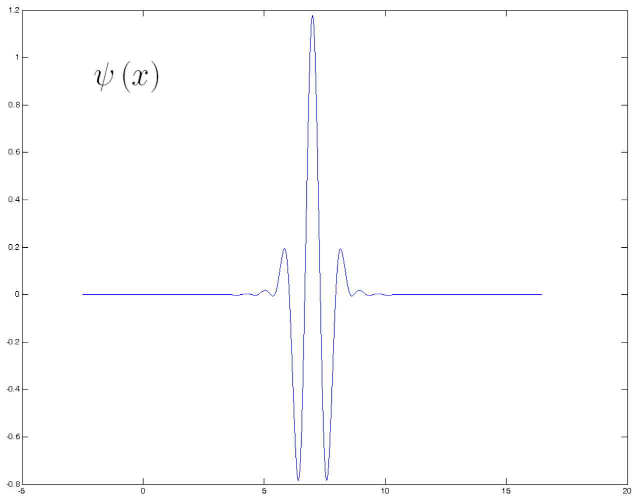

Figure 3. The analysis wavelet for the 33/7 filter bank

V. CONCLUSIONS AND FUTURE WORK

We have designed linear phase biorthogonal filter banks with the purpose of minimizing the time-frequency product of the analyzing wavelet. These designs result in significant improvement over the previous designs in terms of the minimum value of the tf-product. One important advantage of this method is the simplicity in formulation: the filter banks are parametrized by a single parameter. In a recent publication [8], a parametrization of 2-channel linear phase biorthogonal filter banks with two parameters is presented, and it would be interesting to investigate the effect of the addition of a degree of freedom over the time-frequency localization performance of the filter bank. For the purpose of image processing, two-dimensional wavelets can be readily obtained from the wavelets designed by us via an appropriately chosen McClellan's transformation (preferred in 2D filter bank design for many reasons such as preservation of PR property, preservation of the number of zeros at the aliasing frequencies and simplicity) since all the filters we have designed have zero phase response.

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