Performance of the Decorrelating Detector in Correlated Rayleigh Fading

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Abstract — **Consider a code-division multiple-access system in a correlated Rayleigh fading environment in which the fading gains are independent among the users but correlated among the receive diversity branches, and which uses the decorrelating multiuser detector. The branches are combined using maximal-ratio combining, which requires channel estimation. The channel is estimated by means of pilot symbols. From the characteristic function of the detector decision variable, a closed-form expression for the bit error probability (BEP) of this system is obtained. For the case of exponentially correlated branches and equicorrelated signature waveforms, numerical results show that (1) the signature cross-correlation has a stronger effect on the performance than the branch correlation, (2) the BEP steadily decreases with increase in the number of branches, and increases with increase in the number of users, rapidly tending to reach saturation as the number of users becomes large.**

I. Introduction

Although orthogonality among the users' signature waveforms in a code-division multiple-access (CDMA) system [1] is desirable, restrictions on the code length or other system parameters may result in correlated signatures [2, 3]. A multiuser detector, which detects all the users' data together, is effective for such a system since the data streams come from a non-orthogonal multiplex.

Different techniques can be incorporated for multiuser detection. The optimal detector gives good performance but is computationally complex. On the other hand, sub-optimal detectors, like the linear ones, have good performance with low computational complexity. The commonly used linear detectors are the decorrelating detector [2] and the minimum mean-square error (MMSE) detector [3].

The performance of the decorrelating and the MMSE detectors in additive white Gaussian noise (AWGN) has been analyzed in [2] and [3], respectively. The error performance of the decorrelating detector for a CDMA system in independent Rayleigh fading has been numerically evaluated in [4]. The performance of forward and reverse links of a CDMA system with decorrelating detection at the receiver has been studied in [5]. The pairwise error probability of the optimal detector for space-time-coded CDMA in independent Rayleigh fading has been derived in [6]. We focus here on the performance of the decorrelating detector in terms of the bit error probability (BEP) of an user for a CDMA system subject to correlated Rayleigh fading. The fading gains are independent among the users but correlated among the receive diversity branches. The branches are combined using maximal-ratio combining (MRC). Since MRC requires knowledge of the channel, which constitutes the fading gains of the diversity branches of all users, we consider the case when the channel is estimated by means of pilot symbols using both maximum likelihood (ML) and maximum aposteriori probability (MAP) criteria. The characteristic function (c.f.) of the decision variable of an user is derived, and from it a closed-form expression for the BEP is obtained. Numerical results are presented to show the effect of varying system parmeters on the error performance.

II. Channel Model

Consider a synchronous CDMA system over a flat fading channel with K users which uses multiuser detection. The signal is received through L diversity branches. The complex baseband signal received at the lth diversity branch in a symbol interval $[0, T)$ is given by [4] $r_l(t) = \sum_{k=1}^{K} b_k\, c_{k,l}\, s_k(t) + n_l(t)\,, \quad l = 1, \ldots, L$, where $c_{k,l}$ is the random complex fading gain for the kth user and the lth diversity branch, and $s_k(t)$ is the signature waveform of the kth user such that $\int_0^T |s_k(t)|^2\, dt = 1$, and $\int_0^T s_k(t)\, s_{k'}^*(t)\, dt = \rho_{k',k}\,,\ \ k' \neq k,\ \rho_{k',k}$ being the normalized cross-correlation between the signature waveforms $s_{k'}(t)$ and $s_k(t)$, $(\cdot)^*$ denoting the complex conjugate, which we call the *normalized signature cross-correlation*. The transmitted bit of the kth user is b_k ($b_k \epsilon \{-1, 1\}$), and $n_l(t)$ is the AWGN, which is a zero mean complex circular Gaussian white random process with power spectral density $2N_0$ satisfying

$$\mathbf{E}\left[n(t_1)\right] = 0\,, \quad \mathbf{E}\left[n(t_1)\, n(t_2)\right] = 0\,,$$
$$\mathbf{E}\left[n(t_1)\, n^*(t_2)\right] = 2N_0\, \delta(t_1 - t_2)\,, \qquad \forall\, t_1, t_2, \qquad (1)$$

where $\mathbf{E}[\cdot]$ denotes the expectation operator and $\delta(\cdot)$ the Dirac delta function. The noise processes $n_1(t), \ldots, n_L(t)$ are independent among each other and of the fading gains $\{c_{k,l}\}$. The receiver obtains matched filter outputs as

$$y_{k',l} = \int_0^T r_l(t)\, s_{k'}^*(t)\, dt = \sum_{k=l}^{K} b_k\, c_{k,l}\, \rho_{k',k} + \mathcal{N}_{k',l}\,, \quad (2)$$

where $\mathcal{N}_{k',l} = \int_0^T n_l(t) s_{k'}^*(t)\, dt$. It is clear from the statistics of $n_l(t)$ that $\{\mathcal{N}_{k,l}\}$ are zero mean complex circular Gaussian random variables satisfying

$$\begin{aligned} &\mathbf{E}\left[\mathcal{N}_{k,l}\right] = 0\,, \quad \mathbf{E}\left[\mathcal{N}_{k_1,l}\mathcal{N}_{k_2,l}^*\right] = 2N_0\, \rho_{k_1 k_2} \qquad \forall\, k_1, k_2, \\ &\mathbf{E}\left[\mathcal{N}_{k_1,l_1}\mathcal{N}_{k_2,l_2}\right] = 0 \;\; \forall\, k_1, k_2, \\ &\mathbf{E}\left[\mathcal{N}_{k_1,l_1}\mathcal{N}_{k_2,l_2}^*\right] = 0 \qquad l_1 \neq l_2\,, \forall\, k_1, k_2. \end{aligned} \quad (3)$$

Representing (2) in matrix notation we get

$$\mathbf{y}_l = \mathbf{R}\mathbf{C}_l\mathbf{b} + \mathbf{n}_l\,, \qquad l = 1, \ldots, L, \quad (4)$$

where

$$\begin{aligned} &\mathbf{R} = [\rho_{i,j}]_{i,j=1}^{K}\,, \\ &\mathbf{y}_l = [y_{1,l}, y_{2,l}, \cdots, y_{K,l}]^T\,, \\ &\mathbf{C}_l = \mathrm{diag}\left([c_{1,l}, c_{2,l}, \cdots, c_{K,l}]^T\right)\,, \\ &\mathbf{n}_l = [\mathcal{N}_{1,l}, \mathcal{N}_{2,l}, \cdots, \mathcal{N}_{K,l}]^T\,, \\ &\mathbf{b} = [b_1, b_2, \cdots, b_K]^T\,, \end{aligned} \quad (5)$$

$(\cdot)^T$ denoting transpose. Note that $\mathbf{R}$ is the *normalized signature cross-correlation matrix* and $\mathbf{b}$ the *data bit vector*. Denoting the *channel gain vector* for the lth diversity branch as $\mathbf{c}_l = [c_{1,l}, c_{2,l}, \cdots, c_{K,l}]^T$, we can rewrite (4) as

$$\mathbf{y}_l = \mathbf{R}\mathrm{diag}(\mathbf{b})\mathbf{c}_l + \mathbf{n}_l\,, \qquad l = 1, \ldots, L. \quad (6)$$

The diagonal *channel gain matrix* $\mathbf{C}_l$ for the lth branch is given by $\mathbf{C}_l = \mathrm{diag}\,(\mathbf{c}_l)$. The noise vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_L$ are independent complex circular Gaussian random vectors with mean zero and covariance matrix given by

$$\mathbf{E}\left[\mathbf{n}_l\mathbf{n}_l^H\right] = 2N_0\mathbf{R}\,, \quad (7)$$

where $(\cdot)^H$ denotes the Hermitian (conjugate transpose) operator, implying that

$$\mathbf{n}_l \sim \mathcal{CN}\left(\mathbf{0}_K,\, 2N_0\mathbf{R}\right)\,, \quad (8)$$

$\mathbf{0}_K$ denoting the $K \times 1$ vector of zeros. Note that the noise vectors are independent of the channel gain vectors.

We consider the case of correlated Rayleigh fading with the complex fading gains independent among the K users but correlated among the L branches with *channel covariance matrix* $\Omega\mathbf{K}_c$ for each user, where Ω is a positive normalizing factor satisfying

$$\Omega = \frac{1}{L}\sum_{l=1}^{L}\mathbf{E}\left[|c_{k,l}|^2\right] = \frac{1}{L}\mathrm{tr}\left(\Omega\mathbf{K}_c\right)\,, \quad (9)$$

which implies $\mathrm{tr}(\mathbf{K}_c) = L$. Thus if $\rho_{c_{l,m}}$ represents the element in the lth row and mth column of the $L \times L$ *normalized channel covariance matrix* $\mathbf{K}_c$, then the channel gain vectors $\mathbf{c}_1, \ldots, \mathbf{c}_L$ are zero-mean jointly complex circular Gaussian random vectors satisfying

$$\mathbf{E}\left[\mathbf{c}_l\mathbf{c}_m^H\right] = \Omega\rho_{c_{l,m}}\mathbf{I}_K\,, \;\; l = 1, \ldots, L,\; m = 1, \ldots, L, \quad (10)$$

where $\mathbf{I}_K$ denotes the $K \times K$ identity matrix.

III. Channel Estimation and Detection

To design a good detector for a fading environment with receive diversity, we need to combine the received signal vectors $\mathbf{y}_1, \ldots, \mathbf{y}_L$ in (4) using MRC. For this, we need to estimate the channel gain vectors $\mathbf{c}_1, \ldots, \mathbf{c}_L$. We consider the case when the channel estimation is done by means of pilot symbols which are known to the receiver. Denote the estimated channel gain vector for the lth branch by $\hat{\mathbf{c}}_l$ and the corresponding channel gain matrix by $\hat{\mathbf{C}}_l = \mathrm{diag}\,(\hat{\mathbf{c}}_l)$.

Consider the case when M_p *pilot symbol vectors* $\mathbf{b}_{p_1}, \ldots, \mathbf{b}_{p_{M_p}}$ are used, and the ith *pilot symbol matrix* $\mathbf{B}_{p_i}$ is given by $\mathbf{B}_{p_i} = \mathrm{diag}\,(\mathbf{b}_{p_i})$. The ith pilot signal at the lth branch of the receiver is expressed as

$$\mathbf{y}_{p_{i,l}} = \mathbf{R}\mathbf{B}_{p_i}\mathbf{c}_l + \mathbf{n}_{p_{i,l}}\,, \;\; i = 1, \ldots, M_p, \;\; l = 1, \ldots, L, \quad (11)$$

where $\mathbf{n}_{p_i,l}$ is the noise vector at the receiver during the pilot transmission phase, and is independent of the noise vectors $\mathbf{n}_1, \ldots, \mathbf{n}_L$ during the data transmission phase. Moreover, $\left\{\mathbf{n}_{p_{i,l}}\right\}_{i=1,\ldots,M_p,\, l=1,\ldots,L}$ are independent and identically distributed (i.i.d.) complex circular Gaussian random vectors, each having a $\mathcal{CN}\left(\mathbf{0}_K,\, 2N_0\mathbf{R}\right)$ distribution.

Denoting the $K \times L$ matrices $\mathbf{Y}_{p_i}$, $\mathbf{N}_{p_i}$, and $\mathbf{C}$ as

$$\begin{aligned} \mathbf{Y}_{p_i} &= \left[\; \mathbf{y}_{p_{i,1}}, \;\; \cdots, \;\; \mathbf{y}_{p_{i,L}} \;\right], \\ \mathbf{N}_{p_i} &= \left[\; \mathbf{n}_{p_{i,1}}, \;\; \cdots, \;\; \mathbf{n}_{p_{i,L}} \;\right], \\ \mathbf{C} &= \left[\; \mathbf{c}_1, \;\; \cdots, \;\; \mathbf{c}_L \;\right], \end{aligned} \quad (12)$$

we can rewrite (11) as

$$\mathbf{Y}_{p_i} = \mathbf{R}\mathbf{B}_{p_i}\mathbf{C} + \mathbf{N}_{p_i}\,, \quad i = 1, \ldots, M_p. \quad (13)$$

Further, denoting the $KM_p \times L$ matrices $\mathbf{Y}_p$ and $\mathbf{N}_p$ and the $KM_p \times K$ matrix $\mathbf{B}_p$ as

$$\mathbf{Y}_p = \begin{bmatrix} \mathbf{Y}_{p_1} \\ \vdots \\ \mathbf{Y}_{p_{M_p}} \end{bmatrix}, \quad \mathbf{N}_p = \begin{bmatrix} \mathbf{N}_{p_1} \\ \vdots \\ \mathbf{N}_{p_{M_p}} \end{bmatrix}, \quad \mathbf{B}_p = \begin{bmatrix} \mathbf{B}_{p_1} \\ \vdots \\ \mathbf{B}_{p_{M_p}} \end{bmatrix}, \quad (14)$$

(13) can be rewritten as

$$\mathbf{Y}_p = \left(\mathbf{I}_{M_p} \otimes \mathbf{R}\right)\mathbf{B}_p\mathbf{C} + \mathbf{N}_p\,, \quad (15)$$

where $\otimes$ denotes the Kronecker product of matrices.

Let the $KLM_p \times 1$ vectors $\mathbf{y}_p$ and $\mathbf{n}_p$ be given by $\mathbf{y}_p = \mathrm{vec}\left(\mathbf{Y}_p\right)$, $\mathbf{n}_p = \mathrm{vec}\left(\mathbf{N}_p\right)$, and the $KL \times 1$ channel gain vector $\mathbf{c}$ as $\mathbf{c} = \mathrm{vec}\left(\mathbf{C}\right)$. We can now express (15) as

$$\mathbf{y}_p = \left(\mathbf{I}_L \otimes \left(\mathbf{I}_{M_p} \otimes \mathbf{R}\right) \mathbf{B}_p\right) \mathbf{c} + \mathbf{n}_p \,, \tag{16}$$

where, from (7), (8), and (10), $\mathbf{c}$ and $\mathbf{n}_p$ are independent zero-mean complex circular Gaussian random vectors, satisfying

$$\mathbf{E}\left[\mathbf{c}\mathbf{c}^H\right] = \Omega \mathbf{K}_c \otimes \mathbf{I}_K \,, \quad \mathbf{E}\left[\mathbf{n}_p \mathbf{n}_p^H\right] = 2N_0 \mathbf{I}_L \otimes \left(\mathbf{I}_{M_p} \otimes \mathbf{R}\right) \,. \tag{17}$$

Owing to the joint Gaussian statistics of the zero-mean vectors $\mathbf{c}$ and $\mathbf{y}_p$, it can be shown that the MAP estimate of $\mathbf{c}$, which we denote as $\hat{\mathbf{c}}_{MAP}$, is given by

$$\begin{aligned} \hat{\mathbf{c}}_{MAP} &= \Bigg(\Omega \mathbf{K}_c \otimes \mathbf{B}_p^H \left(\mathbf{I}_{M_p} \otimes \mathbf{R}\right) \mathbf{B}_p \\ &\quad + 2N_0 \mathbf{I}_L \otimes \mathbf{I}_K\Bigg)^{-1} \left(\Omega \mathbf{K}_c \otimes \mathbf{B}_p^H\right) \mathbf{y}_p \,. \end{aligned} \tag{18}$$

We can express $\hat{\mathbf{c}}_{MAP}$ as

$$\hat{\mathbf{c}}_{MAP} = \left[\hat{\mathbf{c}}_{1_{MAP}}^T, \cdots, \hat{\mathbf{c}}_{L_{MAP}}^T\right]^T \,. \tag{19}$$

The MAP estimate of the channel matrix for the lth branch is therefore given by $\hat{\mathbf{C}}_{l_{MAP}} = \mathrm{diag}(\hat{\mathbf{c}}_{l_{MAP}})$.

Similarly, the ML estimate of $\mathbf{c}$, which we denote as $\hat{\mathbf{c}}_{ML}$, is given by

$$\begin{aligned} \hat{\mathbf{c}}_{l_{ML}} &= \left[\sum_{i=1}^{M_p} \mathbf{B}_{p_i}^* \mathbf{R} \mathbf{B}_{p_i}\right]^{-1} \left[\sum_{i=1}^{M_p} \mathbf{B}_{p_i}^* \mathbf{y}_{p_i,l}\right] \,, \\ &\quad l = 1, \ldots, L. \end{aligned} \tag{20}$$

From (20), we can express $\hat{\mathbf{c}}_{ML}$ as

$$\begin{aligned} \hat{\mathbf{c}}_{ML} &= \left[\hat{\mathbf{c}}_{1_{ML}}^T, \cdots, \hat{\mathbf{c}}_{L_{ML}}^T\right]^T \\ &= \left(\mathbf{I}_L \otimes \left(\mathbf{B}_p^H \left(\mathbf{I}_{M_p} \otimes \mathbf{R}\right) \mathbf{B}_p\right)^{-1} \mathbf{B}_p^H\right) \mathbf{y}_p \,. \end{aligned} \tag{21}$$

Note that this can also be obtained by putting the condition $(1/L)\mathrm{tr}\left(\Omega \mathbf{K}_c\right) \gg 2N_0$ in (18). The ML estimate of the channel matrix for the lth branch is given by $\hat{\mathbf{C}}_{l_{ML}} = \mathrm{diag}(\hat{\mathbf{c}}_{l_{ML}})$.

Assume that $\mathbf{B}_p$ is chosen such that it satisfies the condition

$$\mathbf{B}_p^H \left(\mathbf{I}_{M_p} \otimes \mathbf{R}\right) \mathbf{B}_p = \sum_{i=1}^{M_p} \mathbf{B}_{p_i}^* \mathbf{R} \mathbf{B}_{p_i} = \beta \mathbf{I}_K \,, \quad \beta > 0. \tag{22}$$

Under condition (22), $\hat{\mathbf{c}}$ is given by

$$\hat{\mathbf{c}} = \begin{cases} \left(\frac{1}{\beta}\left(\mathbf{I}_L + \frac{2N_0}{\beta\Omega} \mathbf{K}_c^{-1}\right)^{-1} \otimes \mathbf{B}_p^H\right) \mathbf{y}_p & \text{for MAP,} \\ \left(\frac{1}{\beta} \mathbf{I}_L \otimes \mathbf{B}_p^H\right) \mathbf{y}_p & \text{for ML.} \end{cases} \tag{23}$$

In other words, we can express the estimate of the channel gain vector $\mathbf{c}$ as $\hat{\mathbf{c}} = \left(\mathbf{A}^T \otimes \mathbf{B}_p^H\right) \mathbf{y}_p$, where the $L \times L$ matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{cases} \frac{1}{\beta}\left(\mathbf{I}_L + \frac{2N_0}{\beta\Omega}\left(\mathbf{K}_c^T\right)^{-1}\right)^{-1} & \text{for MAP,} \\ \frac{1}{\beta} \mathbf{I}_L & \text{for ML.} \end{cases} \tag{24}$$

The estimate of the $K \times L$ *channel matrix* $\mathbf{C}$ can therefore be expressed as

$$\hat{\mathbf{C}} = \left[\ \hat{\mathbf{c}}_1, \ \cdots, \ \hat{\mathbf{c}}_L\ \right] = \mathbf{B}_p^H \mathbf{Y}_p \mathbf{A} \,. \tag{25}$$

Let $\hat{b}_k$ denote the decoded bit of the kth user. In the decorrelating detector, each of the diversity branch signal vectors $\mathbf{y}_1, \ldots, \mathbf{y}_L$ is first decorrelated by premultiplication by $\mathbf{R}^{-1}$ and combined by MRC using the channel estimates $\hat{\mathbf{C}}_1, \ldots, \hat{\mathbf{C}}_L$. The decision is based on the sign of the real part of the resulting vector. Therefore, the decoded data bit vector $\hat{\mathbf{b}}$ is given by

$$\hat{\mathbf{b}} = \mathrm{sgn}\left[\mathrm{Re}\left(\sum_{l=1}^{L} \hat{\mathbf{C}}_l^H \mathbf{R}^{-1} \mathbf{y}_l\right)\right] , \tag{26}$$

where $\hat{\mathbf{b}} = \left[\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_K\right]^T$, and $\mathrm{sgn}[\cdot]$ denotes the signum or sign function.

IV. Error Analysis

Let Γ denote the *average signal-to-noise ratio (SNR) per user per diversity branch* and Γ_p the *average pilot-to-noise ratio (PNR) per user per diversity branch*, given by

$$\Gamma = \frac{\Omega}{2N_0} \,, \quad \Gamma_p = \frac{\beta\Omega}{2N_0} \,, \tag{27}$$

respectively. Substituting (25) in (15) and using condition (22), we get $\hat{\mathbf{C}} = \beta \mathbf{C}\mathbf{A} + \mathbf{B}_p^H \mathbf{N}_p \mathbf{A}$. This can be rewritten as

$$\hat{\mathbf{C}}^T = \beta \mathbf{A}^T \mathbf{C}^T + \mathbf{A}^T \mathbf{N}_p^T \mathbf{B}_p^* \,. \tag{28}$$

Let

$$\begin{aligned} \mathbf{C}^T = \left[\ \mathbf{c}_1', \ \cdots, \ \mathbf{c}_K'\ \right], \ \hat{\mathbf{C}}^T = \left[\ \hat{\mathbf{c}}_1', \ \cdots, \ \hat{\mathbf{c}}_K'\ \right], \\ \mathbf{A}^T \mathbf{N}_p^T \mathbf{B}_p^* = \left[\ \mathbf{e}_1', \ \cdots, \ \mathbf{e}_K'\ \right], \end{aligned} \tag{29}$$

where the $L \times 1$ vectors $\mathbf{c}_k'$ and $\hat{\mathbf{c}'}_k$ are given by

$$\mathbf{c}_k' = [c_{k,1}, c_{k,2}, \ldots, c_{k,L}]^T, \ \hat{\mathbf{c}}_k' = [\hat{c}_{k,1}, \hat{c}_{k,2}, \ldots, \hat{c}_{k,L}]^T, \tag{30}$$

and $\hat{c}_{k,l}$ is the kth element of $\hat{\mathbf{c}}_l$.

We can now express (28) as

$$\hat{\mathbf{c}}_k' = \beta \mathbf{A}^T \mathbf{c}_k' + \mathbf{e}_k' \,, \quad k = 1, \ldots, K, \tag{31}$$

where k is the user index, and $\mathbf{e}_1', \ldots, \mathbf{e}_K'$ are i.i.d. $\mathcal{CN}\left(\mathbf{0}_L, 2N_0\beta\left(\mathbf{A}^T\right)^2\right)$ random vectors, while

$\mathbf{c}'_1, \ldots, \mathbf{c}'_K$ are i.i.d. $\mathcal{CN}(\mathbf{0}_L, \Omega\mathbf{K}_c)$ random vectors, which are independent of $\mathbf{e}'_1, \ldots, \mathbf{e}'_K$. Note that $\mathbf{e}'_k$ represents the channel estimation error for the kth user.

We denote as $g_{k,l}$ the kth element of the $K \times 1$ vector $\mathbf{R}^{-1}\mathbf{n}_l$, implying $g_{k,l} = \left(\mathbf{R}^{-1}\mathbf{n}_l\right)_k$. Let the $L \times 1$ vector $\mathbf{g}'_k$ be given by

$$\mathbf{g}'_k = [g_{k,1}, g_{k,2}, \ldots, g_{k,L}]^T . \tag{32}$$

From (26), the decoded data bit of the kth user can be expressed as $\hat{b}_k = \text{sgn}[D_k]$, where, substituting (6) in (26) and using (32), the kth user's decision variable D_k is given by

$$D_k = \left(\hat{\mathbf{c}}'_k\right)^H (b_k\mathbf{c}'_k + \mathbf{g}'_k) + (b_k\mathbf{c}'_k + \mathbf{g}'_k)^H \hat{\mathbf{c}}'_k , \tag{33}$$

which is a quadratic form in complex Gaussian random variables. Without loss of generality, assume $b_k = 1$. The BEP of the kth user, which we denote as P_{e_k}, is then given by $P_{e_k} = \Pr(D_k < 0)$.

Let $\lambda_{c_1}, \ldots, \lambda_{c_N}$ denote the distinct eigenvalues of $\mathbf{K}_c$, with λ_{c_i} having multiplicity m_i for $i = 1, \ldots, N$. Thus $\sum_{i=1}^N m_i = L$. Note that since $\mathbf{K}_c$ is a positive definite matrix, $\lambda_{c_1}, \ldots, \lambda_{c_N} > 0$. In addition, owing to condition (9), we have $\sum_{i=1}^N m_i\lambda_{c_i} = L$. The c.f. of the quadratic form D_k in (33) can be written in terms of the eigenvalues of $\mathbf{K}_c$ as [7]

$$\begin{aligned}\Psi_{D_k}(\jmath\omega) &= \mathbf{E}\left[e^{\jmath\omega D_k}\right] \\ &= \frac{1}{\prod_{i=1}^N \left(\begin{array}{l} 1 - \jmath\omega(2N_0\Gamma)(2\nu_i) \\ +\omega^2(2N_0\Gamma)^2\left(\mu_{i,k}^2 - \nu_i^2\right)\end{array}\right)^{m_i}} ,\end{aligned} \tag{34}$$

where $\jmath = \sqrt{-1}$, and

$$\nu_i = \begin{cases} \frac{\Gamma_p\lambda_{c_i}^2}{\Gamma_p\lambda_{c_i}+1} & \text{for MAP,} \\ \lambda_{c_i} & \text{for ML,}\end{cases} \tag{35}$$

$$\mu_{i,k} = \begin{cases} \lambda_{c_i}\sqrt{\frac{\Gamma_p\left(\Gamma\lambda_{c_i}+\left(\mathbf{R}^{-1}\right)_{k,k}\right)}{\Gamma\left(\Gamma_p\lambda_{c_i}+1\right)}} & \text{for MAP,} \\ \sqrt{\frac{\left(\Gamma\lambda_{c_i}+\left(\mathbf{R}^{-1}\right)_{k,k}\right)\left(\Gamma_p\lambda_{c_i}+1\right)}{\Gamma_p\Gamma}} & \text{for ML.}\end{cases} \tag{36}$$

Let z denote a complex variable. We can rewrite the c.f. in (34) as

$$\Psi_{D_k}\left(\frac{z}{2N_0\Gamma}\right) = \frac{1}{\prod_{i=1}^N \left(1 - z\Lambda_{i1}^{-1}\right)^{m_i}\left(1 - z\Lambda_{i2}^{-1}\right)^{m_i}} , \tag{37}$$

where $\Lambda_{i1}, \Lambda_{i2}$, $i = 1, \ldots, N$, are the poles of $\Psi_{D_k}(z/(2N_0\Gamma))$, given by

$$\Lambda_{i1} = \frac{1}{\mu_i + \lambda_{c_i}} , \quad \Lambda_{i2} = -\frac{1}{\mu_i - \lambda_{c_i}} . \tag{38}$$

Since $\mu_i > \lambda_{c_i} > 0$, we have $\Lambda_{i1} > 0$ and $\Lambda_{i2} < 0$.

The BEP of the kth user, which is $\Pr(D_k < 0)$, can be obtained from the c.f. of D_k using the inversion theorem [8]. This results in

$$\begin{aligned}P_{e_k} = -\Bigg[&\text{sum of residues of } \frac{\Psi_{D_k}\left(\frac{z}{2N_0\Gamma}\right)}{z} \\ &\text{at poles on left-half } z\text{-plane,} \\ &\text{that is, at } \Lambda_{j2},\ j = 1, \ldots, N\Bigg] .\end{aligned} \tag{39}$$

The residues can be obtained from the derivaties of the c.f. Using Faa di Bruno's formula for the derivatives of a composite function as in [9], we can express P_{e_k} as

$$\begin{aligned}P_{e_k} = &\sum_{j=1}^N \left[\prod_{i=1}^N \left(\frac{\mu_{j,k} - \nu_j}{\mu_{i,k} + \mu_{j,k} + \nu_i - \nu_j}\right)^{m_i}\right] \\ &\times \left[\prod_{\substack{i=1 \\ i\neq j}}^N \left(\frac{\mu_{j,k} - \nu_j}{\mu_{j,k} - \mu_{i,k} + \nu_i - \nu_j}\right)^{m_i}\right] \\ &\times \sum_{\substack{(l_1,\ldots,l_{m_j-1}) \\ 0\le l_1,\ldots,l_{m_j-1}\le m_j-1 \\ l_1+2l_2+\cdots+(m_j-1)l_{m_j-1}=m_j-1}} \prod_{r=1}^{m_j-1} \frac{1}{l_r!}\Bigg[\frac{1}{r} \\ &+ \frac{1}{r}\sum_{i=1}^N \frac{m_i(\mu_{i,k} + \nu_i)^r}{(\mu_{i,k} + \mu_{j,k} + \nu_i - \nu_j)^r} \\ &+ \frac{1}{r}\sum_{\substack{i=1 \\ i\neq j}}^N \frac{m_i(-1)^r(\mu_{i,k} - \nu_i)^r}{(\mu_{j,k} - \mu_{i,k} + \nu_i - \nu_j)^r}\Bigg]^{l_r},\end{aligned} \tag{40}$$

where ν_i is given by (35) and $\mu_{i,k}$ by (36). Thus (40) is a *closed-form expression for the BEP of the kth user* in terms of the distinct eigenvalues $\lambda_{c_1}, \ldots, \lambda_{c_N}$ of the normalized channel covariance matrix $\mathbf{K}_c$.

V. Numerical Results and Conclusion

The BEP of the kth user is computed using (40) for the case of exponential branch correlation, where the element in the lth row and the mth column of the $L \times L$ normalized channel covariance matrix $\mathbf{K}_c$ (see (10)) is given by

$$\rho_{c_{l,m}} = \begin{cases} 1, & \text{if } l = m, \\ \rho_c^{|l-m|}, & \text{if } l \neq m,\end{cases} \tag{41}$$

ρ_c denoting the *normalized branch correlation* and satisfying $0 \le \rho_c < 1$. In this case, all eigenvalues of $\mathbf{K}_c$ are distinct. MAP estimation of the channel is considered. The signature waveforms are taken to be equicorrelated, implying that the element in the ith row and the jth column of the $K \times K$ normalized signature cross-correlation

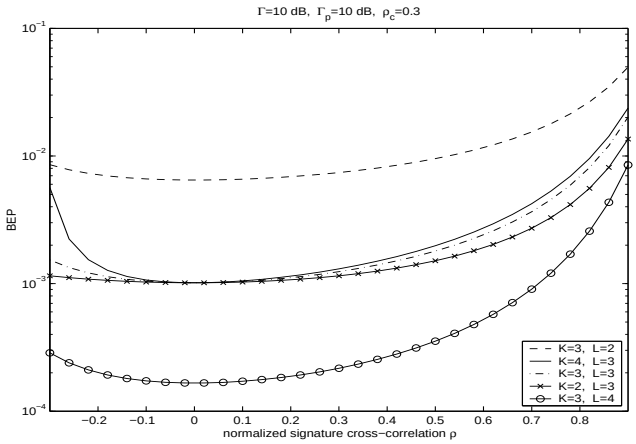

Fig. 1. BEP versus ρ for different values of K and L with $\Gamma = 10$ dB, $\Gamma_p = 10$ dB, and $\rho_c = 0.3$

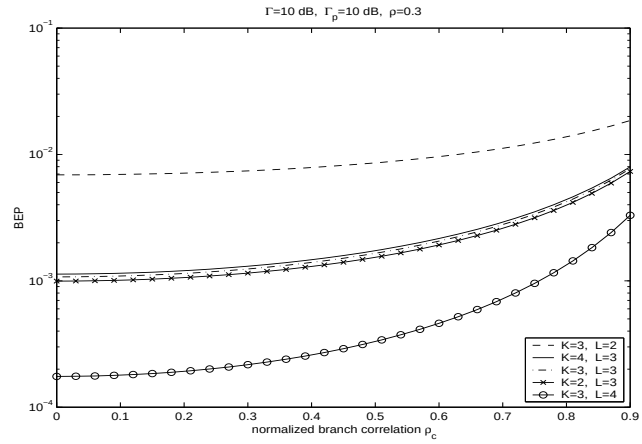

Fig. 2. BEP versus ρ_c for different values of K and L with $\Gamma = 10$ dB, $\Gamma_p = 10$ dB, and $\rho = 0.3$

matrix $\mathbf{R}$ is given by $(\mathbf{R})_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ \rho, & \text{if } i \neq j, \end{cases}$ where ρ is the normalized signature cross-correlation, and satisfies [2, 3] $-1/(K-1) < \rho < 1$. Therefore

$$\left(\mathbf{R}^{-1}\right)_{k,k} = \frac{1 + (K-2)\rho}{(1-\rho)(1+(K-1)\rho)} \tag{42}$$

is the same for all k, and consequently, the BEP P_{e_k} is the same for all users.

Plots of the BEP versus the normalized signature cross-correlation ρ for different values of the number of users K and the number of diversity branches L with average SNR per user per diversity branch $\Gamma = 10$ dB, average PNR per user per diversity branch $\Gamma_p = 10$ dB, and normalized branch correlation $\rho_c = 0.3$ are shown in Fig. 1. We find that the BEP increases with increase of $|\rho|$, with the rate of performance degradation increasing with enhanced signature cross-correlation. It is also seen that an increase in the number of users K increases the BEP while an increase in the number of branches L decreases the BEP, but the change due to L is more significant than that due to K. Fig. 2 gives plots of the BEP versus ρ_c for different values of K and L with $\Gamma = 10$ dB, $\Gamma_p = 10$ dB, and $\rho = 0.3$. A trend similar to that in Fig. 1 is observed. However, although the BEP increases with increase of ρ_c, the effect of ρ_c on the performance is less than that of ρ.

Fig. 3 shows plots of the BEP versus L for different values of K with fixed Γ, Γ_p, ρ and ρ_c. We find that while the BEP steadily decreases with increase in L due to more available diversity, it increases with increase in K and rapidly tends to reach saturation.

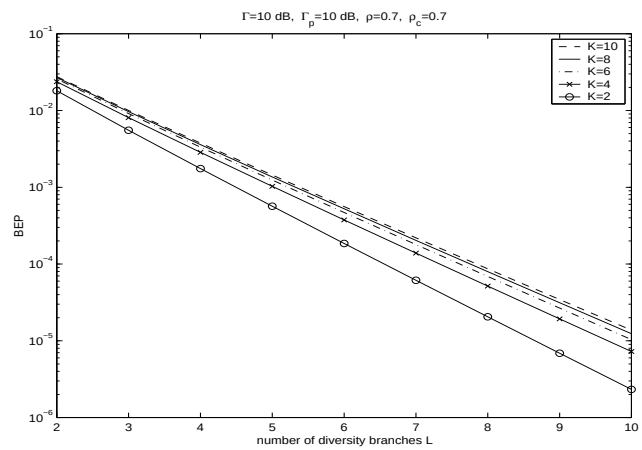

Fig. 3. BEP versus L for different values of K with $\Gamma = 10$ dB, $\Gamma_p = 10$ dB, $\rho = 0.7$, and $\rho_c = 0.7$

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