Low correlation interleaved QAM sequences

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Abstract—A construction for an interleaved selected family, \(T\overline{SQ}_{M^2}\), of low-correlation sequences over the \(M^2\)-QAM constellation, \(M = 2^m, m \geq 3\), is presented. The maximum normalized correlation of the sequences in this family is bounded above by \(\frac{2}{\sqrt[4]{M^2}}\), where \(N\) is the period of the sequences in the family, and where \(a\) ranges from 1.89 in the case of \(64\)-QAM modulation to 1.99 for large \(m\).

These constructions have the lowest-known value of maximum correlation magnitude of any sequence family with the same alphabet.

I. INTRODUCTION

Low-correlation sequences over the QAM alphabet are of interest partly on account of the increasingly common usage of the QAM alphabet as a signaling alphabet and also on account of their ability to carry a larger number of data bits per sequence period in comparison with the one or two bits traditionally associated with BPSK or QPSK sequence families. They also have the potential for permitting variable data transmission on the reverse link of a CDMA system.

Summary of results

We first construct family \(T\overline{SQ}_{M^2}\) with \(M = 2^m, m \geq 4\), which is a family of interleaved selected sequences over the \(M^2\)-QAM constellation. For large \(m\) and large \(N\), the maximum normalized correlation of family \(T\overline{SQ}_{M^2}\) is bounded above by \(1.99\sqrt[4]{N}\), where \(N\) is the period of the sequences in the family. This is less than the maximum normalized correlation of family \(\overline{SQ}_{64}\) [1], which is bounded above by \(2.11\sqrt[4]{N}\).

The results of this paper bring us closer to the Welch lower bound [12] on sequence correlation.

We list the properties of various families of sequences over the \(M^2\)-QAM constellation, \(M = 2^m, m \geq 2\) in Table I. The table also summarizes the results of this paper. Note that the interleaved family \(T\overline{SQ}_{16}\) of [1] has been renamed \(T\overline{SQ}_{04}\) in this paper.

II. BACKGROUND

A. \(M^2\)-QAM constellation

The \(M^2\)-QAM constellation is the set\n\[
\{a + ib \mid -M + 1 \leq a, b \leq M - 1, \ a, b \text{ odd}\}.
\]

When \(M = 2^m\), this constellation can be equivalently defined as [6],[11]
\[
\left\{(1+i) \left(\sum_{k=0}^{m-1} 2^k i^a_k\right) \mid a_i \in \mathbb{Z}_4\right\}.
\]

The equivalence between the two representations follows from noting that an odd number, \(x\), in the range \([-M+1, M-1]\), \(M = 2^m\), can be expressed as
\[
x = \sum_{k=0}^{m-1} 2^k(-1)^{x_k}, \ x_k \in \mathbb{Z}_2
\]
and the relation
\[
(-1)^{x_i+x_j} + i(-1)^{x_j} = (1+i)x_i+2x_j, \ x_i, x_j \in \mathbb{Z}_2.
\]

The second representation for \(M^2\)-QAM, \(M = 2^m\), suggests that a sequence over \(M^2\)-QAM can be constructed using a collection of sequences over \(\mathbb{Z}_4\), of size \(m\), and we adopt this approach in this paper.

B. Galois Rings

Let \(\mathbb{Z}_n\) denote the ring of integers modulo \(n\). In this paper, our primary interest is in the ring \(\mathbb{Z}_4 = \{0,1,2,3\}\).

Galois rings [7] are Galois extensions of the prime ring \(\mathbb{Z}_p\). \(R = GR(4, r)\) will denote a Galois extension...
This is often referred to as the “2-adic expansion” of z. Modulo-2 reduction of z is denoted by \( \overline{z} \). It can be shown that every element \( z \in R \) can uniquely be expressed as

\[
z = a + 2b, \quad a, b \in \mathcal{T}.
\]

This is often referred to as the “2-adic expansion” of \( z \). Modulo-2 reduction of \( z \) is denoted by \( \overline{z} \). It can be shown that \( \alpha = \overline{z} \) is a primitive element in \( \mathbb{F}_2 \).

More details on Galois rings can be found in [7],[4],[5],[9].

C. Sequence correlation

The periodic correlation between two complex-valued sequences, \( \{s(j, t)\} \) and \( \{s(k, t)\} \), at time shift \( \tau \) is defined as

\[
\theta_{s(j), s(k)}(\tau) = \sum_{t=0}^{N-1} s(j, t + \tau) \overline{s(k, t)}
\]

where \( 0 \leq \tau \leq (N - 1) \)

with \( (t + \tau) \) computed modulo \( N \). This form of correlation is also referred to as even-periodic correlation to differentiate it from other forms of correlation between two sequences.

The maximum correlation parameter for a family of sequences is defined to be

\[
\theta_{\text{max}} := \max \left\{ |\theta_{s(j), s(k)}(\tau)| \right\} \quad \text{either } j \neq k \text{ or } \tau \neq 0.
\]

However, as proposed in [1], we make two changes to the maximum correlation parameter. The first change recognizes that when a user is assigned multiple spreading sequences, a bank of correlators is used at the receiver end and the autocorrelation between two such sequences at zero shift does not interfere with the self-synchronization capability of the family. Accordingly, the maximum non-trivial correlation magnitude of a sequence family is given the modified definition:

\[
\theta_{\text{max}} := \max \left\{ |\theta_{s(j), s(k)}(\tau)| \right\} \quad \text{either } s(j, t), s(k, t) \text{ have been assigned to distinct users or } \tau \neq 0.
\]

The second change arises from energy considerations. To make a fair comparison between QAM and PSK families, it is required that the correlation magnitude be normalized to take into account the larger energy of the QAM sequence families. We use \( \overline{\theta}_{\text{max}} \) to denote the maximum correlation magnitude after energy normalization and this is used as the basis for comparison across signal constellations.

<table>
<thead>
<tr>
<th>Family</th>
<th>Constellation</th>
<th>Period</th>
<th>Family Size</th>
<th>Data rate</th>
<th>Energy</th>
<th>( \overline{\theta}_{\text{min}} )</th>
<th>( \overline{\theta}_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CQ_{16} [1]</td>
<td>16-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{2} + 1 )</td>
<td>4</td>
<td>10N</td>
<td>0.4N</td>
<td>1.80N</td>
</tr>
<tr>
<td>SQ_{16} [1]</td>
<td>16-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{2} + 1 )</td>
<td>3</td>
<td>10N</td>
<td>0.8N</td>
<td>1.61N</td>
</tr>
<tr>
<td>TSQ_{16} [1]</td>
<td>16-QAM</td>
<td>( N = 2(2^r - 1) )</td>
<td>( \frac{N}{2} + 1 )</td>
<td>3</td>
<td>10N</td>
<td>2N</td>
<td>1.41N</td>
</tr>
<tr>
<td>CQ_{64} [1]</td>
<td>64-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{4} + 1 )</td>
<td>6</td>
<td>42N</td>
<td>0.10N</td>
<td>2.33N</td>
</tr>
<tr>
<td>SQ_{64} [1]</td>
<td>64-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{4} + 1 )</td>
<td>4</td>
<td>42N</td>
<td>0.19N</td>
<td>2.11N</td>
</tr>
<tr>
<td>TSQ_{64} [1]</td>
<td>64-QAM</td>
<td>( N = 2(2^r - 1) )</td>
<td>( \frac{N}{4} + 1 )</td>
<td>4</td>
<td>42N</td>
<td>0.19N</td>
<td>1.89N</td>
</tr>
<tr>
<td>CQ_{256} [1]</td>
<td>256-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{8} + 1 )</td>
<td>8</td>
<td>170N</td>
<td>0.023N</td>
<td>2.65N</td>
</tr>
<tr>
<td>SQ_{256} [1]</td>
<td>256-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{8} + 1 )</td>
<td>5</td>
<td>170N</td>
<td>0.047N</td>
<td>2.41N</td>
</tr>
<tr>
<td>TSQ_{256} [1]</td>
<td>256-QAM</td>
<td>( N = 2(2^r - 1) )</td>
<td>( \frac{N}{8} + 1 )</td>
<td>5</td>
<td>170N</td>
<td>0.12N</td>
<td>1.95N</td>
</tr>
<tr>
<td>CQ_{1024} [1]</td>
<td>1024-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{16} + 1 )</td>
<td>10</td>
<td>682N</td>
<td>0.006N</td>
<td>2.82N</td>
</tr>
<tr>
<td>SQ_{1024} [1]</td>
<td>1024-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{16} + 1 )</td>
<td>6</td>
<td>682N</td>
<td>0.012N</td>
<td>2.58N</td>
</tr>
<tr>
<td>TSQ_{1024} [1]</td>
<td>1024-QAM</td>
<td>( N = 2(2^r - 1) )</td>
<td>( \frac{N}{16} + 1 )</td>
<td>6</td>
<td>682N</td>
<td>0.029N</td>
<td>2.01N</td>
</tr>
<tr>
<td>CQ_{M2} [1]</td>
<td>( M^2 )-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{M^2} + 1 )</td>
<td>2m</td>
<td>( \frac{M^2 - 1}{2} N )</td>
<td>( \frac{M^2 - 1}{2} N )</td>
<td>3N</td>
</tr>
<tr>
<td>SQ_{M2} [1]</td>
<td>( M^2 )-QAM</td>
<td>( N = 2^r - 1 )</td>
<td>( \frac{N}{M^2} + 1 )</td>
<td>m + 1</td>
<td>( \frac{M^2 - 1}{2} N )</td>
<td>( \frac{M^2 - 1}{2} N )</td>
<td>2.76N</td>
</tr>
<tr>
<td>TSQ_{M2}</td>
<td>( M^2 )-QAM</td>
<td>( N = 2(2^r - 1) )</td>
<td>( \frac{N}{M^2} + 1 )</td>
<td>m + 1</td>
<td>( \frac{M^2 - 1}{2} N )</td>
<td>( \frac{M^2 - 1}{2} N )</td>
<td>1.99N</td>
</tr>
</tbody>
</table>
D. Family $A$

Family $A$ is an asymptotically optimal family of quaternary sequences (i.e., over $\mathbb{Z}_4$) discovered independently by Sole [10] and Boztaş, Hammons and Kumar [2], [3]. A detailed description of their correlation properties appears in [3].

Let $\{\gamma_i\}_{i=1}^{2^r}$ denote $2^r$ distinct elements in $\mathbb{T}$, i.e., we have the alternate expression $\mathbb{T} = \{\gamma_1, \gamma_2, \ldots, \gamma_{2^r}\}$. There are $2^r + 1$ cyclically distinct sequences in Family $A$, each of period $2^r - 1$. The following representation for sequences in Family $A$ is used in this paper:

$$s_i(t) = T((1 + 2\gamma_i)\xi^t), \quad 1 \leq i \leq 2^r,$$

$$s_{2^r+1}(t) = 2T(\xi^t).$$

The following theorem summarizes the correlation properties of sequences from Family $A$.

**Theorem 2.1:** Consider two sequences from Family $A$ defined as

$$s(a, t) = T((1 + 2a)\xi^t) \quad \text{and} \quad s(b, t) = T((1 + 2b)\xi^t), \quad a, b \in \mathbb{T}.$$ 

Then

$$\theta_{s(a), s(b)}(\tau) = \begin{cases} 2^r - 1, & a = b, \tau = 0 \\ -1, & a \neq b, \tau = 0 \\ -1 + \Gamma(1)\tau^{-\tau}(z), & \tau \neq 0 \end{cases}$$

where

$$z = a + \frac{a + b}{y} + \frac{1}{\sqrt{y}} + 2\gamma$$

with

$$y = \xi^r + 1 + 2\sqrt{\xi^r}$$

and $\gamma$ chosen to ensure that $z \in \mathbb{T}$.

We refer the reader to [3], [13] for a detailed proof.

E. Prior constructions in the literature

In this subsection, we briefly summarize the properties of the five sequence families proposed by Anand and Kumar [1].

Canonical family $CQ_{M^2}$ has period $N$, normalized maximum-correlation parameter $\theta_{\text{max}}$ bounded above by $\lesssim a\sqrt{N}$, where $a$ ranges from 1.6 in the 16-QAM case to 3.0 for large $M$. In a CDMA setting, each user is enabled to transfer $2m$ bits of data per period of the spreading sequence.

Selected family $SQ_{M^2}$ has a lower value of $\theta_{\text{max}}$ but permits only $(m + 1)$-bit data modulation. Family $ISQ_{M^2}$ has period $N$, normalized maximum-correlation parameter $\theta_{\text{max}}$ bounded above by $\lesssim a\sqrt{N}$, where $a$ ranges from 1.61 in the 16-QAM case to 2.76 for large $M$.

The interleaved selected 16-QAM sequence family $\mathcal{I}S\mathcal{Q}_{16}$ ($\mathcal{I}Q_{16}$ in [1]) has $\theta_{\text{max}} \lesssim \sqrt{2\sqrt{N}}$ and supports 3-bit data modulation.

The properties of the above mentioned families of sequences have been summarized in Table I.

The remaining two families are over a quadrature-PAM (Q-PAM) subset of size $2M$ of the $M^2$-QAM constellation. Family $P_{2M}$ has a lower value of $\theta_{\text{max}}$ in comparison with family $SQ_{M^2}$, while still permitting $(m + 1)$-bit data modulation. Interleaved family $\mathcal{I}P_{8}$, over the 8-ary Q-PAM constellation, permits 3-bit data modulation and achieves the Welch lower bound [12] on $\theta_{\text{max}}$.

III. INTERLEAVED SEQUENCES OVER $M^2$-QAM

In this section, we assume $m \geq 4$. The special case of $m = 3$ i.e. 64-QAM is discussed in the next section.

Let $\{\delta_0 = 0, \delta_1, \delta_2, \ldots, \delta_{m-1}\}$ be elements from $\mathbb{F}_q$ such that $tr(\delta_k) = 1, \forall k \geq 1$. Set $H = \{\delta_0, \delta_1, \ldots, \delta_{m-1}\}$. Let $G = \{g_k\}$ be the largest subset of $\mathbb{F}_q$ having the property that

$$g_k + \delta_p \neq g_l + \delta_q, \quad g_k, g_l \in G, \quad \delta_p, \delta_q \in H,$$

unless $g_k = g_l$ and $\delta_p = \delta_q$. Then the corresponding Gilbert-Varshamov and Hamming bounds [8] on the size of $G$ are given by

$$\frac{2^r}{1 + \binom{m-1}{1} + \binom{m-1}{2}} \leq |G| \leq \frac{2^r}{1 + \binom{m-1}{1}} \quad (1)$$

A. Subspace-based construction for $G$ and $H$ [1]

Given constellation parameter $m$, let $2^l$ denote the smallest power of 2 greater than $(m - 1)$, i.e., $l$ is defined by

$$2^{l-1} < (m - 1) \leq 2^l. \quad (6)$$

We refer to the integer $l$ as the subspace-size exponent (sse) associated with the constellation parameter (c-p) $m$. Thus $l$ lies in the range $0 \leq l \leq (r - 1)$. Let $\mu$ denote the function that, given c-p $m$ in the range $1 \leq m \leq 2^{r-1} + 1$, maps $m$ to the corresponding sse $l$ given above, i.e., $\mu(m) = l$.

Treating $\mathbb{F}_q$ as a vector space over $\mathbb{F}_2$ of dimension $r$, let $W_{r-1}$ denote the subspace of $\mathbb{F}_q$ of dimension $(r - 1)$ corresponding to the elements of trace $= 0$. Let $W_l$ denote a subspace of $W_{r-1}$ having dimension $l$. Let $\zeta$ be an element in $\mathbb{F}_q$ having trace 1 and let $V_l$ denote the subspace $V_l = W_l + \zeta$ of size $2^{l+1}$.

Noting that every element in the coset $W_l + \zeta$ of $W_l$ has trace 1, we select as the elements $\{\delta_k\}_{k=1}^{m-1}$ to be used in the construction of Family $\mathcal{I}S\mathcal{Q}_{M^2}$, an arbitrary collection of $(m - 1) \leq 2^l$ elements selected from the set $W_l + \zeta$.

Next, we partition $W_{r-1}$ into the $2^{r-l-1}$ cosets $W_l + g$ of $W_l$. With each coset, we associate a distinct user.
\[ s(g, \kappa, t) = \begin{cases} 
(1 + i) \left( \sum_{k=1}^{m-1} 2m-k-1 \ell u_k(t)(-1) \kappa_k + 2m-1 \ell u_0(t) \right) t^{\kappa_0}, & t \text{ even} \\
(1 + i) t \left( \sum_{k=1}^{m} 2m-k \ell u_k(t)(-1) \kappa_k + 2m-1 \ell u_0(t) \right) t^{\kappa_0}, & t \text{ odd} 
\end{cases} \]

\[ s(g, 0, t) = \begin{cases} 
(1 + i) \left( \frac{1}{2} \ell u_1(t) + \frac{3}{2} \ell u_2(t) + 2 \ell u_3(t) + 4 \ell u_4(t) + 8 \ell u_6(t) \right), & t \text{ even} \\
(1 + i) t \left( \frac{1}{2} \ell u_1(t) + \frac{3}{2} \ell u_2(t) + 2 \ell u_3(t) + 4 \ell u_4(t) - 8 \ell u_6(t) \right), & t \text{ odd} 
\end{cases} \]

To this user, we assign the coefficient set \( \{ g, g + \delta_1, g + \delta_2, \ldots, g + \delta_l \} \). The coefficients \( \{ g + \delta_k \}_{k=1}^{m} \) belong to the coset \( W_1 + (g + \zeta) \) of \( W_1 \). Thus, in general, each user is assigned \( m \) coefficients, with one coefficient \( g \), belonging to the coset \( W_1 + g \) of \( W_1 \) lying in \( W_{r-1} \) and the remaining drawn from the coset \( W_1 + g + \zeta \) of \( W_1 \). Since \( V_i = W_i \cup (W_i + \zeta) \), all \( m \) coefficients taken together belong to the coset \( V_i + g \) of \( V_i \). Note that

\[ V_i + g = V_i + g' \]

implies

\[ \{ W_i + g \} \cup \{ W_i + \zeta + g \} = \{ W_i + g' \} \cup \{ W_i + \zeta + g' \}. \]

But this is impossible since \( g, g' \) belong to different cosets of \( W_i \) and \( g, g' \) have trace zero, whereas, \( tr(\zeta) = 1 \). It follows that the coefficient sets of distinct users belong to different cosets of \( V_i \) and are hence distinct.

Let \( G \) be the set of all such coset representatives of \( W_i \) in \( W_{r-1} \). Since each user is associated to a unique coset representative, the number of users is given by \( G = 2^{r-l-1} \). When combined with (6), we obtain

\[ \frac{2^r}{4(m-1)} < |G| \leq \frac{2^r}{2(m-1)}. \]

Thus, the size of \( G \) is at most a factor of 4 smaller than the best possible suggested by the Hamming bound (1). The reader is referred to [1] for more details on the construction of \( G \) and \( H \).

### B. Sequence definition

We now describe our interleaved construction of the sequences in the family \( ISQ_{M^2} \).

Let \( \{ \tau_1, \tau_2, \ldots, \tau_{m-1} \} \) be a set of non-zero, distinct time-shifts with \( \{ 1, \alpha, \alpha^2, \ldots, \alpha^{m-1} \} \) being a linearly independent set. Let \( \kappa = (\kappa_0, \kappa_1, \ldots, \kappa_{m-1}) \in \mathbb{Z}_4 \times \mathbb{F}_2^{m-1} \). Family \( ISQ_{M^2} \) is then defined as follows:

\[ ISQ_{M^2} = \left\{ \{ s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_4 \times \mathbb{F}_2^{m-1} \} \mid g \in G \right\} \]

so that each user is identified by an element of \( G \). Each user is assigned the collection

\[ \{ s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_4 \times \mathbb{F}_2^{m-1} \} \]

of sequences with the \( \kappa \)-th sequence given by (2) and where

\[ u_0(t) = T([1 + 2g]t^\xi), \quad u_k(t) = T([1 + 2(g + \delta_k)]t^{\xi + \tau_k}), \quad k = 1, 2, \ldots, m - 1. \]

We refer to the element \( g \) as the ground coefficient. Note that given the ground coefficient and the set \( \{ \delta_1, \ldots, \delta_{m-1} \} \), the set of coefficients used by a user are uniquely determined. The elements \( \{ \delta_k \} \) provide a selection of the component sequences that leads to lower correlation values.

Within the subset of the sequences assigned to a particular user, the sequences corresponding to \( \kappa = 0 \) are termed basic sequences. Basic sequences have a simpler representation and correlations involving basic sequences turn out to be representative of the general case.

An alternate representation for the \( \kappa \)-th sequence for family \( ISQ_{M^2} \) is provided in (3). Examples of basic
sequences for \( m = 4 \) and \( m = 5 \) are given in (4) and (5) respectively.

\[
s(g, \kappa, t) = \begin{cases} 
(1 + i) \left( t^{u_2(t)}(-1)^{\kappa_2} + 2 t^{u_1(t)}(-1)^{\kappa_1} + 4 t^{u_0(t)} \right) v_0^\kappa, & t \text{ even} \\
(1 + i) \left( t^{u_2(t)}(-1)^{\kappa_2} - 2 t^{u_0(t)} + 4 t^{u_1(t)}(-1)^{\kappa_1} \right) v_0^\kappa, & t \text{ odd}
\end{cases}
\] (8)

\[ N : m \]

C. Properties

The main properties of family \( ISQ_{M^2} \), \( M = 2^m, m \geq 4 \) are summarized in Table I. The correlation values can be computed by using Theorem 2.1, and by using the fact that \( \theta_{u_0,v_0} \) is at right angles to \( \theta_{u_1,v_1} \) and \( \theta_{u_2,v_2} \) [1], where \( u_i \) and \( v_i \) are the family \( A \) sequences assigned to the two users.

The family size is given by (7). Note from (1) that this can potentially be improved by a different construction of the set \( G \).

The number \( N \) of times an element from the \( M^2 \) QAM constellation occurs in sequences of large period can be bounded as:

\[
|N - N + 1| \leq \frac{M^2 - 1}{M^2} \sqrt{N + 1},
\]

i.e., the sequences in family \( ISQ_{M^2} \) are approximately balanced.

Due to space constraints, we do not give a proof of these properties in this paper; the proofs would be presented in a detailed version later.

IV. INTERLEAVED SEQUENCES OVER 64-QAM

The comments made in the starting of the previous section are applicable here. The choice of \( \delta_k \) and \( \tau_k \) and the construction of \( G \) and \( H \) is the same as in the previous section. However, the sequence definition is different from the general case discussed before.

A mathematical expression for family \( ISQ_{64} \) is provided below.

\[
ISQ_{64} = \left\{ \{ s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_4 \times \mathbb{F}_2 \} \mid g \in G \right\}.
\]

Each user is thus assigned the set

\[
\{ s(g, \kappa, t) \mid \kappa \in \mathbb{Z}_4 \times \mathbb{F}_2 \}
\]

of sequences with the \( \kappa \)-th sequence given by (8) and where

\[
\begin{align*}
u_0(t) &= T([1 + 2g] \xi^t), \\
u_1(t) &= T([1 + 2(g + \delta_1)] \xi^{t + \tau_1}), \\
u_2(t) &= T([1 + 2(g + \delta_2)] \xi^{t + \tau_2}).
\end{align*}
\]

The properties of sequence period, energy, family size and balance are similar to the corresponding properties for the general case discussed in section III, i.e., family \( ISQ_{M^2} \) with \( M = 2^m, m \geq 4 \).

However, the normalized minimum squared Euclidean distance between all sequences assigned to a user is given by

\[
\bar{d}_{\min}^2 \approx 0.19N
\]

and for large values of \( N \), the maximum normalized correlation of the family is bounded as

\[
\bar{d}_{\max} \lesssim \frac{3\sqrt{2}}{2} \sqrt{N} \approx 1.89\sqrt{N}.
\]

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