On The Use of Dirac Delta Distribution in Transformation of Random Variables

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***Abstract*—In this paper, we present proofs with typical examples, for the use of Dirac delta distribution as an easier tool to evaluate the probability density function (PDF) of the transformed random variables (r.v.'s) and also, we alternatively prove this by deriving it through characteristic function.**

I. Introduction

A continuous linear functional on a set of testing functions is called a distribution [3], [2]. Dirac delta belong to the class of singular distributions and is defined as

$$\int_{-\infty}^{\infty} \phi(x)\delta(x-x_0)dx \triangleq \phi(x_0) \tag{1}$$

The integral in (1) has no meaning in classical sense, but it is only a symbolic notation and (1) says that $\delta(x-x_0)$ assigns $\phi(x)$ a number $\phi(x_0)$.

Let supp $\phi(x)$ and supp $\delta(x-x_0) = x_0$ are the supports of the test function and the delta distribution respectively, then [2]

$$\text{supp } [\phi(x)\delta(x-x_0)] = \text{supp } \phi(x) \cap \text{supp } \delta(x-x_0) \tag{2}$$

Now consider the following expression.

$$\int_{-\infty}^{z} \phi(x)\delta(x-x_0)dx = \begin{cases} \phi(x_0) & \text{for } z > x_0 \\ 0 & \text{for } z < x_0. \end{cases} \tag{3}$$

From (2) we see that for $z < x_0$, $\delta(x-x_0)$ and $\phi(x)$ has no common support and hence $\phi(x)$ is mapped to 0. For the case $z > x_0$ the only common support point is x_0 and $\phi(x)$ is mapped to $\phi(x_0)$.

The transformation property of Dirac delta distribution is given by [2]

$$\delta[g(x)] = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|g'(x_i)|} \tag{4}$$

with $|g'(x_i)| \neq 0$ and x_i's are the simple roots of the equation $g(x) = 0$.

Example 1:

(a)

$$\delta(ax+b) = \frac{1}{|a|}\delta\left(x + \frac{b}{a}\right) \tag{5}$$

(b)

$$\delta\left[x^2 - x_0^2\right] = \frac{1}{2x_0}\left[\delta(x-x_0) + \delta(x+x_0)\right] \tag{6}$$

It is emphasized that equalities in (5) and (6) says LHS and RHS are equal only in distributional sense.

II. Transformation of Random Variables

A. Single Function of Random Variables

Let X be a r.v. with PDF $f_X(x)$, then the PDF $f_Y(y)$ of the transformed r.v. $Y = g(X)$ is given by [4],

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|} \tag{7}$$

where x_i's are the simple roots of the equation $g(x) - y = 0$. (7) can be written as

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\left[\sum_{i=1}^{n} \frac{\delta(x-x_i)}{|g'(x_i)|}\right] dx \tag{8}$$

By comparing (4) and (8) we get the following important result.

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\delta\left[g(x) - y\right] dx \tag{9}$$

(9) can be extended to one function of many r.v.'s as

$$f_Y(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x})\delta\left[g(\mathbf{x}) - y\right] d\mathbf{x} \tag{10}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is joint PDF of n r.v.'s $X_1, X_2, \cdots, X_n$ and $d\mathbf{x} = dx_1 \cdots dx_n$. The result in (10) can be succinctly written as

$$f_Y(y) = E\Big\{\delta(g(\mathbf{x}) - y)\Big\} \tag{11}$$

where E is an expectation operator. We demonstrate the usefulness of this approach in the following examples.

Example 2: Let $Y = X^2$. Using (9) we get

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\delta\left[x^2 - y\right] dx \tag{12}$$

and by (6)

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x) \left\{ \frac{1}{2\sqrt{y}} \left[\delta(x+\sqrt{y}) + \delta(x-\sqrt{y}) \right] \right\} dx \tag{13}$$

Thus by (1)

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left[f_X(-\sqrt{y}) + f_X(\sqrt{y}) \right] \tag{14}$$

Example 3: Let $Z = (X+Y)^2$. The PDF of Z is given by

$$f_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \delta \left[(x+y)^2 - z \right] dxdy \tag{15}$$

Using (4), $\delta \left[(x+y)^2 - z \right]$ can be written as

$$\delta \left[(x+y)^2 - z \right] = \frac{1}{2\sqrt{z}} \left\{ \delta \left[x+y-\sqrt{z} \right] + \delta \left[x+y+\sqrt{z} \right] \right\} \tag{16}$$

This simplifies (15) as

$$f_Z(z) = \frac{1}{2\sqrt{z}} \left\{ \int_{-\infty}^{\infty} f_{XY} \left(\sqrt{z} - y, y \right) dy \right\} + \frac{1}{2\sqrt{z}} \left\{ \int_{-\infty}^{\infty} f_{XY} \left(-\sqrt{z} - y, y \right) dy \right\} \tag{17}$$

Example 4: Let $Y = \max(X_1, X_2, \ldots, X_n)$ where $X_1, X_2, \ldots, X_n$'s are n r.v.'s with their joint PDF $f_{\mathbf{X}}(\mathbf{x})$. The PDF of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x}) \delta \left[\max(x_1, x_2, \ldots, x_n) - y \right] d\mathbf{x} \tag{18}$$

As

$$\max(x_1, x_2, \ldots, x_n) = \begin{cases} x_1 & \text{for } x_1 > x_i, i = 2, \cdots, n \\ x_2 & \text{for } x_2 > x_i, i = 1, 3, \cdots, n \\ \vdots & \vdots \\ x_n & \text{for } x_n > x_i, i = 1, \cdots, n-1 \end{cases}$$

(18) simplifies to

$$f_Y(y) = \int_{-\infty}^{\infty} \left\{ q_1(x_1) \right\} \delta \left[x_1 - y \right] dx_1 + \cdots + \int_{-\infty}^{\infty} \left\{ q_n(x_n) \right\} \delta \left[x_n - y \right] dx_n \tag{19}$$

where $q_1(x_1) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_1} f_{\mathbf{X}}(\mathbf{x}) dx_2 \cdots dx_n$ and $q_n(x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_n} f_{\mathbf{X}}(\mathbf{x}) dx_1 \cdots dx_{n-1}$. By using (3) in (19) we get

$$f_Y(y) = q_1(y) + q_2(y) + \cdots + q_n(y) \tag{20}$$

If $X_1, X_2, \ldots, X_n$'s are all i.i.d.'s with marginal PDF $f_X(x)$ and cumulative distribution function $F_X(x)$ then (20) results in

$$\begin{aligned} f_Y(y) &= n f_X(y) \underbrace{\int_{-\infty}^{y} f_X(x) dx \cdots \int_{-\infty}^{y} f_X(x) dx}_{n-1 \text{ terms}} \\ &= n f_X(y) [F_X(y)]^{n-1} \end{aligned} \tag{21}$$

B. Functions of Several Random Variables

1) Two Functions of Two Random Variables: Let X and Y be two r.v.'s with joint PDF $f_{XY}(x,y)$. The joint PDF of the transformed r.v.'s $U = g(X,Y)$ and $V = h(X,Y)$ is given by

$$f_{UV}(u,v) = \sum_i \frac{1}{\left| J(x_i, y_i) \right|} f_{XY}(x_i, y_i) \tag{22}$$

where $\left| J \right|$ represents Jacobian of the transformation [4]. (22) can be written as

$$f_{UV}(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \mathbf{T}_1 dxdy \tag{23}$$

where $\mathbf{T}_1 = \sum_i \frac{1}{\left| J(x_i, y_i) \right|} \delta(x - x_i, y - y_i)$. By using the direct product property, we can replace $\delta(x - x_i, y - y_i)$ in $\mathbf{T}_1$ with $\delta(x - x_i)\delta(y - y_i)$.

It can also be shown that

$$f_{UV}(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \mathbf{T}_2 dxdy \tag{24}$$

where $\mathbf{T}_2 = \delta \left[g(x,y) - u \right] \delta \left[h(x,y) - v \right]$. In other words we say that $\mathbf{T}_1 = \mathbf{T}_2$. To prove this let us assume that the transformation is one-to-one. Let point (x_1, y_1) in $x-y$ plane gets transformed to (u_1, v_1) in $u-v$ plane. Also let $\phi(x,y)$ be a 2-D test function. Then $\delta(x - x_1)\delta(y - y_1)$ maps $\phi(x,y)$ to $\phi(x_1, y_1)$ and $\delta(u - u_1)\delta(v - v_1)$ maps $\phi(u,v)$ to $\phi(u_1, v_1)$, i.e.,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x,y) \delta(x - x_1) \delta(y - y_1) dxdy = \phi(x_1, y_1) \tag{25}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(u,v) \delta(u - u_1) \delta(v - v_1) dudv = \phi(u_1, v_1) \tag{26}$$

$\phi(u_1, v_1)$ gets transformed to $\phi(x_1, y_1)$ as given below

$$\phi(u_1, v_1) \longrightarrow \phi(x_1, y_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x,y) \delta \left[g(x,y) - u_1 \right] \times \delta \left[h(x,y) - v_1 \right] \left| J \right| dxdy \tag{27}$$

where $\left| J \right| = \left| \frac{\partial(g,h)}{\partial(x,y)} \right|$ is the Jacobian of the transformation. Thus from (27) and (25) it can be deduced that

$$\delta \left[g(x,y) - u_1 \right] \delta \left[h(x,y) - v_1 \right] = \frac{1}{\left| J \right|} \delta(x - x_1) \delta(y - y_1) \tag{28}$$

If the mapping is many-to-one then the summation $\mathbf{T}_1$ in (23) results.

Example 5: Let $Z = \max(X,Y)$ and $W = \min(X,Y)$. The joint PDF $f_{ZW}(z,w)$ of Z and W is given by

$$f_{ZW}(z,w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \delta \left[\max(x,y) - z \right] \times \delta \left[\min(x,y) - w \right] dxdy \tag{29}$$

As

$$\max(x,y) = \begin{cases} x & \text{for } x > y, \\ y & \text{for } x < y. \end{cases}$$

and

$$\min(x,y) = \begin{cases} x & \text{for } x < y, \\ y & \text{for } x > y. \end{cases}$$

(29) simplifies to

$$f_{ZW}(z,w) = \int_{-\infty}^{z} f_{XY}(z,y)\delta(y-w)dy + \int_{-\infty}^{z} f_{XY}(x,z)\delta(x-w)dx \quad (30)$$

By using (3), for $z < w$, two terms in (30) are zeros. For $z > w$ (30) results in

$$f_{ZW}(z,w) = f_{XY}(z,w) + f_{XY}(w,z) \quad (31)$$

2) n Functions of n Random Variables: The result in (24) can also be extended to n r.v.'s. Let $X_1, X_2, \cdots, X_n$ be n r.v.'s with joint PDF $f_{\mathbf{X}}(\mathbf{x}) = f_{X_1X_2\cdots X_n}(x_1, x_2, \cdots, x_n)$ and let the functions of these n r.v.'s be defined as

$$\begin{aligned} Y_1 &= g_1(X_1, X_2, \ldots, X_n) \\ \vdots &= \vdots \\ Y_n &= g_n(X_1, X_2, \ldots, X_n) \end{aligned} \quad (32)$$

The joint PDF of n r.v.'s $Y_1, Y_2, \cdots, Y_n$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x})\delta\left[g_1(\mathbf{x} - y_1)\right] \cdots \times \delta\left[g_n(\mathbf{x} - y_n)\right] d\mathbf{x} \quad (33)$$

Example 6: Let n i.i.d r.v.'s $X_1, \cdots, X_n$ having joint PDF $f_{\mathbf{X}}(\mathbf{x})$ be transformed to a new set of $n-1$ r.v.'s $Y_1 = \frac{X_2}{X_1}, Y_2 = \frac{X_3}{X_1}, \cdots, Y_{n-1} = \frac{X_n}{X_1}$ with $X_1 \neq 0$. Using (33), the joint PDF of Y_i's can be found to be

$$f_{\mathbf{Y}}(\mathbf{y}) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1 \text{ terms}} f_{\mathbf{X}}(\mathbf{x})\delta\left[y_1 - \frac{x_2}{x_1}\right] \cdots \times \delta\left[y_{n-1} - \frac{x_n}{x_1}\right] dx_1 dx_2 \cdots dx_n \quad (34)$$

As X_i's are independent,

$$f_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{\infty} f_{X_1}(x_1) \int_{-\infty}^{\infty} f_{X_2}(x_2)\delta\left(y_1 - \frac{x_2}{x_1}\right) dx_2 \cdots \times \int_{-\infty}^{\infty} f_{X_n}(x_n)\delta\left(y_{n-1} - \frac{x_n}{x_1}\right) dx_n dx_1 \quad (35)$$

Using (5), as X_i's are identical with PDF, say $f_X(x)$, (35) simplifies to

$$f_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{\infty} |x|^n f_X(x) f_X(y_1 x) \times \cdots \times f_X(y_{n-1}x) dx \quad (36)$$

III. Relation to Characteristic Function

Let $\Phi_Y(\omega)$ be the characteristic function of a r.v. Y is defined as $\Phi_Y(\omega) = E\{e^{j\omega Y}\}$. The PDF $f_Y(y)$ can be obtained using Fourier inversion formula as

$$f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_Y(\omega) e^{-j\omega y} d\omega \quad (37)$$

Using (37), PDF of the transformed r.v. $Y = g(X)$ can be obtained by replacing characteristic function $\Phi_Y(\omega)$ with $E\left\{e^{j\omega g(X)}\right\}$. Thus

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega[g(x)-y]} d\omega\right] dx \quad (38)$$

Using the relation

$$\delta\left[g(x) - y\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega[g(x)-y]} d\omega \quad (39)$$

in (38) we get (9). This can be viewed as an alternative proof of the statement (9). This can be extended to n functions of n r.v.'s. The joint PDF of transformed r.v.'s $\mathbf{Y} = Y_1, \cdots, Y_n$ using its joint characteristic function $\Phi_{\mathbf{Y}}(\boldsymbol{\omega}) = \Phi_{\mathbf{Y}}(\omega_1, \cdots, \omega_n)$ can be written as

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Phi_{\mathbf{Y}}(\boldsymbol{\omega}) \exp\left\{-j\sum_{i=1}^{n} \omega_i y_i\right\} d\boldsymbol{\omega} \quad (40)$$

where $d\boldsymbol{\omega} = d\omega_1, \cdots, d\omega_n$.

As $\Phi_{\mathbf{Y}}(\boldsymbol{\omega}) = E\left\{\exp \sum_{i=1}^{n} j\omega_i g(x_i)\right\}$, (40) results in

$$f_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(\mathbf{x}) \times \exp\left(j\sum_{i=1}^{n} \omega_i \Big[g(x_i) - y_i\Big]\right) d\boldsymbol{\omega}\right] d\mathbf{x} \quad (41)$$

On expanding the exponential in to individual products and by using the relation in (39), (41) finally results in (33).

IV. Conclusion

In this paper we discussed a method to evaluate PDF of transformed r.v.'s using Dirac delta distribution. This method is simpler than the existing one in the way that it serves as an alternative to the the evaluation of Jacobian by not explicitly evaluating it, and this is demonstrated with the help of examples. The proof with the help of characteristic function emphasizes the use of Dirac delta as distribution in finding PDFs. The authors would thank the anonymous reviewers for their comments and suggestions.

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