Signal Matched Non-uniform Filter Bank

Sanjay Nalbalwar         S. D. Joshi         R. K. Patney
Research Scholar,        Professor,         Professor,
EE Dept., IIT Delhi,      EE Dept., IIT Delhi, EE Dept., IIT Delhi,
nalbalwar_sanjayan@yahoo.com  sdjoshi@ee.iitd.ernet.in  rkpatney@ee.iitd.ernet.in

Abstract

In this paper we present a non-uniform filter bank (NUFB) matched to a given signal. To obtain matched M-channel NUFB, first, we choose the decimation set having M-down sampling/decimation factors for which perfect reconstruction NUFB exist and then using novel approach proposed in this paper, M-channel signal matched analysis bank is estimated. The outputs of all filters at the analysis side of proposed filter bank are mutually as well as across various channels are uncorrelated. By using well established theory of multirate filter bank, M-channel NUFB matched to signal is obtained. The analysis filters obtained in this paper are found to be FIR whereas synthesis filters are FIR or IIR. Finally, the result of simulation for deterministic case is appended in the table. Filter bank obtained in this fashion will be useful to compress code or represent the signal or image in the best possible manner.

1. Introduction

Multirate analysis/synthesis filter bank systems are conventionally used to decompose signals into separate frequency bands. The most frequently studied cases of filter bank systems are one with equal sampling factors, called uniform filter bank (UFB) system. The M-channel UFB is shown in fig. 1. Perfect reconstruction (PR) property of this filter bank has been extensively studied [1-3] and also multirate models matched the signal are well studied in [4]. This system splits the input signal \( x(n) \) into subband signals \( x_k(n) \) which have equal bandwidths [1-3].

![Figure 1: M-Channel Maximally Decimated Filter Bank.](image)

In some applications such as audio, speech and image analysis and coding, the information desired lies in particular frequency band if we are analyzing this signal with uniform filter banks then it may happen that the band of interest is divided into two or more bands, in this situation instead of UFB if choose NUFBs then we divide

the input spectrum in such bands so that the band of interest will be a single band so only that particular output is processed specially [5]. NUFB is widely used in audio and speech processing application and offers considerable flexibility in decomposition of frequency bands. As stated earlier, multirate 2–band and M-band uniform filter model matched to a signal or statistics is studied in [6] but such model for NUFB was not available. This paper is attempt towards the finding of signal matched NUFB.

In this paper, we are presenting novel approach to find non-uniform filter bank (NUFB) matched to a given signal. The result of simulation for deterministic case is appended in the table. Similarly, results for given correlation case can be obtained. Filter bank we have considered in this paper is NUFB with integer sampling rate, however, our proposed approach can easily be extended to rational sampling filter bank. Filter bank obtained in this fashion will be useful to compress code or represent the signal or image in the best possible manner.

The outline of the paper will be as follows. In section-2 gives the relevant theory of UFB and how to convert NUFB into equivalent UFB such that well developed theory UFB can be used to find the synthesis side of NUFB. In section-3 we have presented the novel method of finding matched NUFB to a given signal. In section-4, there will be discussion on simulation results. Last section ends with conclusion.

2. M-Channel Maximally Decimated Filter Bank

A uniformly decimated M-band bank where \( H_0(z), H_1(z), \ldots, H_{M-1}(z) \) are transfer functions of analysis bank filters, \( F_0(z), F_1(z), \ldots, F_{M-1}(z) \) are the transfer functions of synthesis bank filters is shown in Fig.1. A signal \( x(n) \) is passed through all the filters in the analysis bank simultaneously, in fig. 1 so the input signal \( x(n) \) is split into M subbands by analysis filters \( H_i(z), 0 \leq i \leq M -1 \), and then decimated by M to produce subband signal \( x_i(k), 0 \leq i \leq M -1 \). At synthesis end, subband signals \( x_i(k) \) ’s are interpolated by M and then passed through synthesis filters \( F_i(z) \)’s and
then added to produce the reconstructed signal $\hat{x}(n)$. The subband signals $x_i(k)'s$ can be mathematically written as

$$x_i(n) = \sum_{n=\infty}^{\infty} x(k)h_i(Mn-k) \quad i=0,1,\ldots M-1$$

(1)

The reconstructed signal $\hat{x}(n)$ is given as

$$\hat{x}(n) = \sum_{i=0}^{M-1} \sum_{k=\infty}^{\infty} x_i(k) f_i(n-kM)$$

(2)

In case of perfect reconstruction filter bank $\hat{x}(n)$ will be will be equal to shifted and scaled version of $x(n)$.

For M-channel maximally decimated FB the analysis filters can be written in terms of Type-I, M-component polyphase matrix $E(z)$ [5] as:

$$H_0(z) = E_{00}(z^M) + z^{-1}E_{01}(z^M) + \cdots + z^{-(M-1)}E_{0M-1}(z^M)$$

$$H_0(z) = E_{00}(z^M) + z^{-1}E_{11}(z^M) + \cdots + z^{-(M-1)}E_{1M-1}(z^M)$$

$$H_{M-1}(z) = E_{M-1,0}(z^M) + z^{-1}E_{M-1,1}(z^M) + \cdots + z^{-(M-1)}E_{M-1,M-1}(z^M)$$

(3)

Similarly, the synthesis filters can be written in terms of Type-II, M-component polyphase matrix $R(z)$ [5] as:

$$F_0(z) = R_{00}(z^M) + z^{-1}R_{01}(z^M) + \cdots + z^{-(M-1)}R_{0M-1}(z^M)$$

$$F_1(z) = R_{11}(z^M) + z^{-1}R_{11}(z^M) + \cdots + z^{-(M-1)}R_{1M-1}(z^M)$$

$$F_{M-1}(z) = R_{M-1,0}(z^M) + z^{-1}R_{M-1,1}(z^M) + \cdots + z^{-(M-1)}R_{M-1,M-1}(z^M)$$

$$F_{M-1}(z) = R_{M-1,0}(z^M) + z^{-1}R_{M-1,1}(z^M) + \cdots + z^{-(M-1)}R_{M-1,M-1}(z^M)$$

(4)

$$R(z)E(z) = cz^{-k}$$

If (5) is satisfied then the output of FB [1-3] is

$$\hat{x}(n) = cx(n-k)$$

(6)

where $c$ is some constant and $k$ is integer.

and the output is constant times delayed version of input signal.

So far we have discussed the case of uniform M-band maximally decimated filter bank. Now we give brief relevant review of results associated with NUFB, one whose channel decimation rates need not all be equal. Due to unequal decimation factors procedure for obtaining PR is different than UFB.

$$Figure 2: Polyphase Representation.$$
Now, to obtain the polyphase decomposition of such nonuniform FB we will use the extended polyphase matrix as given by [1]. \( E_{ik}(z) \) and \( R_{ik}(z) \) are \( i^{th} n_i(z) \)-fold Type-I and Type-II polyphase components of \( H_k(z) \) and \( F_k(z) \) respectively and are given as

\[
H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{ik}(z^{-n})
\]

(7)

\[
F_k(z) = \sum_{l=0}^{M-1} z^{-l} R_{ik}(z^{-n})
\]

(8)

To obtain the polyphase representation, redraw NUFB as an equivalent uniform maximally decimated system [3-4]. Define \( N \) to be the least common multiple (LCM) of the \( n_i \) Then for fig.5a equivalent uniform filter structure is given in fig.5b with \( P_k = N/n_i \)

\[ a(n) \rightarrow H_k(n) \rightarrow b_k \rightarrow F_k(n) \rightarrow f(n) \]

Figure 5a: One channel of NUFB

Then using the uniform structure we can develop the polyphase representation as given in fig. 5c. Where \( E(z) \) and \( R(z) \) are given as follows:

\[
E(z) = [E_t(z), E_{k-1}(z), \ldots E_0(z)]^T
\]

(9)

\[
R(z) = [R_0(z), R_{k-1}(z), \ldots R_0(z)]
\]

(10)

By using PR relation (5) synthesis polyphase matrix \( R(z) \) is obtained and in turn synthesis filters of NUFB are estimated.

\[ a(n) \rightarrow H_{k0}(n) \rightarrow h_k(n) \rightarrow b_k \rightarrow F_{k0}(n) \rightarrow f(n) \]

Figure 5b: Equivalent N-channel UFB Fig. 5a

\[ a(n) \rightarrow E_0(n) \rightarrow E_0(n) \rightarrow b_k \rightarrow F_0(n) \rightarrow f(n) \]

Figure 5c: Polyphase representation of Figure 5b

3. Estimation of Filters of A Signal Matched NUFB

In this section, we are proposing a novel method for estimating the filters of NUFB for a given signal case. In subsection-1 we have given method for estimation of analysis filters. Output of all these filters are not only uncorrelated across the channel but also there is no correlation among the channels. In subsection-2 method for obtaining the corresponding synthesis filters is given

3.1. Estimation of filters of NUFB in case of deterministic signal

Let us consider that M-channel NUFB having decimation set \( \{n_0, n_1, \ldots, n_{M-1}\} \). Let \( x(n) \) be a given signal for which we want to find signal matched NUFB. The approach proposed here is, first, given signal \( x(n) \) is divided into M-subsequences as shown in fig 6.

\[ x(n) \rightarrow Z^n \rightarrow R_1 \rightarrow \ldots \rightarrow R_{M-1} \rightarrow x(n) \]

Figure 6: Decomposition of a given sequence \( x[n] \) (q=addition of all delay elements)

Each subsequence \( x_i(n) \) is obtained by shifting \( x(n) \) first by \( I_i \) and then shifted sequence is downsampled by \( n_i \). That is,

\[ x_i(n) = x(n, n-l_i) \quad i = 0, 1, \ldots, M-1 \]

The shift \( I_i \) may be different for different subsequences. This shift and downsampling factors are selected in such a way that every sample must go through one and only one branch, which equivalent to saying that

\[ mn_i - l_i \neq kn_j - l_j \quad \text{for} \ i \neq j \quad \text{any choice of} \ m \text{ and } k \].

After getting M-subsequences in this manner, M filters one corresponding to each are obtained using Least square approach in the following manner.

\[
e_i(n) = x(n, n-l_i) - \sum_{k=1}^{p} h_i(k)x(n, n-l_i - k)
\]

(11)

Where \( h_i(n) \) is a filter corresponding to \( i^{th} \) subsequence.

By writing the equation (11) in vector form we get:

\[
e_i = x_i - X_i h_i^T
\]

(12)

where
\[ \begin{bmatrix} e_i(0) \\ e_i(1) \\ \vdots \\ e_i(N-1) \end{bmatrix}, \quad x_i = \begin{bmatrix} x(-l_i) \\ x(n_i - l_i) \\ \vdots \\ x((N-1)n_i - l_i) \end{bmatrix} \]

\( N \) is an integer constant equal to no. of samples in the given signal

\[ X_i = \begin{bmatrix} x(-l_i) & x(-l_i - 2) & \cdots & x(-l_i - p) \\ x(n_i - l_i) & x(n_i - l_i - 2) & \cdots & x(n_i - l_i - p) \\ \vdots & \vdots & \ddots & \vdots \\ x((N-1)n_i - l_i) & x((N-1)n_i - l_i - 2) & \cdots & x((N-1)n_i - l_i - p) \end{bmatrix} \]

Least square estimate of the filter coefficients of filter \( h_i(n) \) is given as:

\[ h_i = \text{inv}(X_i^*X_i) \cdot X_i^*x_i \quad 0 \leq i \leq M - 2 \]

The last filter \( h_{M-1}(n) \) is estimated from its earlier \((M-1)\) filters because which is carrying the remainig information. Generally this remaining information is correlated, to uncorrelate this, prediction filter is estimated which is used while estimating the last filter. Though outputs across the filters \( h_0(n), h_1(n), \ldots, h_{M-1}(n) \) are uncorrelated but there is correlation among the outputs of these filters. This correlation is removed in the following manner.

To remove the correlation between \( h_0(n) \) and \( h_1(n) \), remove the output of \( h_0(n) \) from the input signal and apply resultant signal as a input to \( h_1(n) \), this can seen as:

\[ \tilde{h}_1(n) = (\delta(n) - h_0(n)) \cdot h_1(n) \]

Similarly, we can find remaining filters in the same manner:

\[ \tilde{h}_2(n) = (\delta(n) - h_0(n)) \cdot (\delta(n) - h_1(n)) \cdot h_2(n) \]

\[ \tilde{h}_{M-2}(n) = (\delta(n) - h_0(n)) \cdot (\delta(n) - h_1(n)) \cdots (\delta(n) - h_{M-3}(n)) \cdot h_{M-2}(n) \]

\[ \tilde{h}_{M-1}(n) = (\delta(n) - h_0(n)) \cdot (\delta(n) - h_1(n)) \cdots (\delta(n) - h_{M-2}(n)) \]

Thus, for a maximally decimated analysis bank consists of M channels. First channel consists of highpass filter \( \tilde{h}_0(n) \) with bandwidth \( \pi / \eta_1 \), next \( M-2 \) channels consists of band pass filters \( \tilde{h}_1(n), \tilde{h}_2(n), \ldots, \tilde{h}_{M-2}(n) \) with bandwidths \( \pi / \eta_1, \pi / \eta_2, \ldots, \pi / \eta_{M-2} \) and that last channels corresponds to lowpass filter \( \tilde{h}_{M-1}(n) \) with bandwidth \( \pi / \eta_{M-1} \). All these filters are real coefficients filters.

### 3.2. Design of M-channel NUBF Perfect Reconstruction Biorthogonal Filter Bank with IIR Synthesis Filters

Consider analysis / synthesis filter bank structure as shown in Fig.3. From analysis filter bank having \( \tilde{h}_0(n), \tilde{h}_1(n), \ldots, \tilde{h}_{M-2}(n), \tilde{h}_{M-1}(n) \) as the analysis filters, a polyphase decomposition matrix \( \tilde{E}(z) \) is formed by first converting this analysis filter bank into equivalent uniform analysis banks and then by using PR relation as in (5) corresponding synthesis polyphase matrix \( \tilde{R}(z) \) is obtained. Then by using relation (4) synthesis filters are derived. It is observed that corresponding synthesis filters are FIR/IIR filters.

#### 4. Simulation Results

The proposed method is applied on 1/f signal with 0.65 a Hurst exponent. Based on the proposed theory, three filters of analysis bank: \( h_0, h_1, \) & \( h_2 \) having decimation factors \( \{2, 4, 4\} \) are estimated from the signal. By using filter bank theory corresponding synthesis filters \( f_0, f_1, \) & \( f_2 \) are estimated. Analysis filters and corresponding synthesis filters for nonuniformly decimated 3-channel filter bank are appended in Table-1. Spectrums of estimated filters are shown in Fig 7.

#### 5. Conclusions

In this paper, first we have given the novel approach to estimate the NUBF matched to a signal. First analysis filters are estimated from the given signal based on decimation set. As the estimated analysis filter bank has non uniform bands, the resulting synthesis filters are obtained by converting this NU analysis bank into uniform analysis bank by using well established theory. In this paper we have considered the case of integer sampling rate filter bank however the same approach can be used to estimate the rational sampling rate filter bank. This model finds application in the area of speech and mage processing. We have presented simulation results on 1/f signal. The resulting analysis filters are found to be FIR whereas synthesis filters are found to be FIR/IIR filters. Filter bank obtained in this fashion will be useful to compress code or represent the signal or image in the best possible manner.
References


Table: 1 Analysis and synthesis filters matched to a given

<table>
<thead>
<tr>
<th>Input Signal</th>
<th>No. of samples &amp; value of H</th>
<th>Estimated Filters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/f signal</td>
<td>1000, 0.65</td>
<td>h0 = [1.0000 -0.3104 -0.5206]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>h1 = [1.0000 -1.3404 1.4359 2.0204 -2.1171]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>h2 = [1.0000 0.4582 0.1282 0.0203 0.1766]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>f0 = [-1.8 - 0.56 - 2.4 1.2 0.3 0.72 - 0.68]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>f1 = [0.77 0.24 -0.22 0.055 -0.049 0.014 -0.023]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>f2 = [1 0.32 2.6 0.98 0.062 0.53 -0.89]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>det (E(z)) = [2.2 0 0 0 -0.16]</td>
</tr>
</tbody>
</table>

Figure 7: Magnitude responses of analysis filters first (row) and synthesis filters (second row)