Effect of Channel Correlation on the Performance of Space-Time Coded Systems

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Abstract — **For a space-time coded system in a correlated Rayleigh fading environment, we consider two receiver structures: (1) a suboptimum one in which the imperfect channel state information (CSI) is used in place of perfect CSI in a perfect CSI optimum receiver, (2) the optimum receiver in which the imperfect CSI is used in the maximum likelihood sense. Channel estimation is done by the use of a pilot code matrix prior to data reception. In the cases of both the receivers, the system performance is analyzed in terms of the pairwise error probability.**

I. Introduction

For a multiple-input multiple-output wireless communication system using space-time block coding [1, 2], the coherent detection of the space-time codes requires the channel state information (CSI). The channel matrix can estimated from known pilot code vectors. However, in practice, the CSI obtained is imperfect [3, 4], and replacing the exact channel matrix by its imperfect estimate in the decision variable processor at the receiver does not give an optimum decision criterion in the maximum likelihood (ML) sense. We consider in this paper two receiver structures [5]: (1) a suboptimum one in which the imperfect CSI is used in place of perfect CSI in a perfect CSI ML receiver, (2) the optimum receiver in which the imperfect CSI is used in the ML sense. In the cases of both the receivers, we analyze the system performance in terms of the pairwise error probability (PEP).

II. Model

Consider a wireless communication system with N transmit and M receive antennas using space-time block coding [1, 2]. An $N \times L$ code matrix is used to transmit $N \times 1$ code vectors over L symbol intervals. Channel estimation prior to data reception is done by using a sequence of L_p pilot code vectors to obtain an estimate of the $M \times N$ channel matrix $\mathbf{H}$. The $M \times 1$ complex baseband signal vector received at time index l by the M antennas after matched filtering is given by [1]

$$\mathbf{r}_l = \mathbf{H}\mathbf{c}_l + \mathbf{n}_l\,, \quad l = 1, \ldots, L, \tag{1}$$

where $\mathbf{c}_l$ denotes the the $N \times 1$ code vector transmitted from the N antennas, $\mathbf{H}$ the $M \times N$ random *channel matrix*, and $\mathbf{n}_l$ the $M \times 1$ additive white Gaussian noise vector. The noise vectors for different time indices l are independent and identically distributed (i.i.d.) complex circular Gaussian random vectors, each $\mathbf{n}_l$ having a $\mathcal{CN}\left(\mathbf{0}_{M\times 1}, 2N_0\mathbf{I}_M\right)$ distribution, where $\mathbf{I}_M$ denotes the $M \times M$ identity matrix, and $2N_0$ is the noise power spectral density. The channel matrix is independent of the noise.

Owing to flat fading, the channel matrix $\mathbf{H}$ is assumed to be constant over the time indices which span the pilot transmission phase followed by the encoded data transmission phase. We consider the case of *correlated Rayleigh fading with only receive correlation*, implying that the columns of $\mathbf{H}$ are independent while the rows are correlated, such that, if $\mathbf{H}$ is expressed as

$$\mathbf{H} = [\mathbf{h}_1, \cdots, \mathbf{h}_N]\,, \tag{2}$$

then $\mathbf{h}_1, \ldots, \mathbf{h}_N$ are i.i.d. complex circular Gaussian random vectors, each having a $\mathcal{CN}\left(\mathbf{0}_{M\times 1}, \mathbf{K}\right)$ distribution, where $\mathbf{K}$ denotes the *receive correlation matrix*.

Channel estimation is done by using a sequence of L_p pilot code vectors $\mathbf{c}_{p_1}, \ldots, \mathbf{c}_{p_{L_p}}$. The $N \times L_p$ pilot code matrix $\mathbf{C}_p$ given by

$$\mathbf{C}_p = \left[\ \mathbf{c}_{p_1},\quad \mathbf{c}_{p_2},\quad \cdots,\quad \mathbf{c}_{p_{L_p}}\ \right]. \tag{3}$$

During the pilot transmission phase, we receive

$$\mathbf{r}_{p_l} = \mathbf{H}\mathbf{c}_{p_l} + \mathbf{n}_{p_l}\,, \quad l = 1, \ldots, L_p, \tag{4}$$

where $\mathbf{n}_{p_1}, \ldots, \mathbf{n}_{p_{L_p}}$ are i.i.d. complex circular Gaussian random vectors, each having a $\mathcal{CN}\left(\mathbf{0}_{M\times 1}, 2N_0\mathbf{I}_M\right)$ distribution.

Denoting the received pilot signal matrix $\mathbf{R}_p$ and the pilot noise matrix $\mathbf{N}_p$ as

$$\mathbf{R}_p = \left[\ \mathbf{r}_{p_1},\quad \mathbf{r}_{p_2},\quad \cdots,\quad \mathbf{r}_{p_{L_p}}\ \right],$$

$$\mathbf{N}_p = \left[\ \mathbf{n}_{p_1},\quad \mathbf{n}_{p_2},\quad \cdots,\quad \mathbf{n}_{p_{L_p}}\ \right],$$

we can rewrite (4) in the form

$$\mathbf{R}_p = \mathbf{H}\mathbf{C}_p + \mathbf{N}_p\,. \tag{5}$$

Since the entries of $\mathbf{H}$ are Gaussian and the additive noise is Gaussian, the minimum mean square estimate (MMSE) of $\mathbf{H}$ is linear in $\mathbf{R}_p$. The least squares estimate (LSE) of $\mathbf{H}$ is always linear in $\mathbf{R}_p$ irrespective of the statistics of $\mathbf{H}$. Therefore, the estimate of $\mathbf{H}$ can be expressed in the form

$$\begin{aligned}\hat{\mathbf{H}} &= \left[\hat{\mathbf{h}}_1, \cdots, \hat{\mathbf{h}}_N\right] \\ &= \mathbf{R}_p\mathbf{Q}\,, \end{aligned} \tag{6}$$

where

$$\mathbf{Q} = \begin{cases} \mathbf{C}_p^H\mathbf{E}\left[\mathbf{H}^H\mathbf{H}\right]\left(\mathbf{C}_p\mathbf{C}_p^H\mathbf{E}\left[\mathbf{H}^H\mathbf{H}\right] + 2N_0M\mathbf{I}_N\right)^{-1} \\ \text{for MMSE} \\ \mathbf{C}_p^H\left(\mathbf{C}_p\mathbf{C}_p^H\right)^{-1} \quad \text{for LSE,} \end{cases} \tag{7}$$

$\mathbf{E}[\cdot]$ denoting the expectation operator and $(\cdot)^H$ the Hermitian (conjugate transpose) operator.

Consider the case when orthogonal pilot codes are used for channel estimation. This gives the condition

$$\mathbf{C}_p\mathbf{C}_p^H = \beta\mathbf{I}_N\,, \quad \beta > 0. \tag{8}$$

This condition also implies $L_p \geq N$. Further, let $\mathrm{tr}(\mathbf{K}) = M\Omega$, such that

$$\mathbf{E}\left[\mathbf{H}^H\mathbf{H}\right] = \mathrm{tr}(\mathbf{K})\mathbf{I}_N = M\Omega\mathbf{I}_N\,. \tag{9}$$

Substituting (8) and (9) in (7), we get

$$\mathbf{Q} = \frac{\rho}{\beta}\mathbf{C}_p^H\,, \tag{10}$$

where

$$\rho = \begin{cases} \frac{\Omega\beta}{\Omega\beta+2N_0} & \text{for MMSE,} \\ 1 & \text{for LSE.} \end{cases} \tag{11}$$

Thus, from (5) and (6) we get

$$\hat{\mathbf{H}} = \rho\mathbf{H} + \frac{\rho}{\beta}\mathbf{N}_p\mathbf{C}_p^H\,. \tag{12}$$

III. Performance of a Suboptimum Receiver

Let $\mathcal{C}_{N,L}$ denote the set of $N \times L$ code matrices used for transmission. Suppose that the code matrix

$$\mathbf{C}_1 = \left[\ \mathbf{c}_{1,1},\quad \mathbf{c}_{2,1},\quad \cdots,\quad \mathbf{c}_{L,1}\ \right] \tag{13}$$

is transmitted after the channel has been estimated. At the receiver, we choose the code matrix

$$\hat{\mathbf{C}}_1 = \left[\ \hat{\mathbf{c}}_{1,1},\quad \hat{\mathbf{c}}_{2,1},\quad \cdots,\quad \hat{\mathbf{c}}_{L,1}\ \right] \tag{14}$$

using the minimum distance rule, which results in a suboptimum decision criterion given by

$$\begin{aligned}\hat{\mathbf{C}}_1 &= \arg\Bigg\{\min_{\left[\ \mathbf{x}_1,\ \cdots,\ \mathbf{x}_L\ \right]\in\mathcal{C}_{N,L}} \\ &\qquad \sum_{l=1}^{L}\left\|\mathbf{r}_l - \hat{\mathbf{H}}\mathbf{x}_l\right\|^2\Bigg\}, \end{aligned} \tag{15}$$

where $\|\cdot\|$ denotes the L_2-norm or Euclidean norm of a vector. From (1), we have

$$\mathbf{r}_l = \hat{\mathbf{H}}\mathbf{c}_{l,1} + \left(\mathbf{H} - \hat{\mathbf{H}}\right)\mathbf{c}_{l,1} + \mathbf{n}_l\,, \quad l = 1, \ldots, L. \tag{16}$$

Defining the $M \times 1$ vector $\mathbf{v}_l$ as

$$\mathbf{v}_l \triangleq \left(\mathbf{H} - \hat{\mathbf{H}}\right)\mathbf{c}_{l,1} + \mathbf{n}_l\,, \tag{17}$$

(16) can be written in the form

$$\mathbf{r}_l = \hat{\mathbf{H}}\mathbf{c}_{l,1} + \mathbf{v}_l\,. \tag{18}$$

Suppose the transmitted code matrix $\mathbf{C}_1$ is wrongly decoded as the code $\mathbf{C}_2$ at the receiver, where

$$\mathbf{C}_2 = \left[\ \mathbf{c}_{1,2},\quad \mathbf{c}_{2,2},\quad \cdots,\quad \mathbf{c}_{L,2}\ \right]\,. \tag{19}$$

The PEP is given by

$$\begin{aligned}P(\mathbf{C}_1 \to \mathbf{C}_2) &= \Pr\Bigg\{\sum_{l=1}^{L}\left\|\hat{\mathbf{H}}(\mathbf{c}_{l,1} - \mathbf{c}_{l,2}) + \mathbf{v}_l\right\|^2 \\ &\qquad < \sum_{l=1}^{L}\|\mathbf{v}_l\|^2\Bigg\}\,. \end{aligned} \tag{20}$$

To obtain the PEP, we focus on the decision variable $\mathcal{D}$, given by

$$\mathcal{D} = \sum_{l=1}^{L}\left\|\hat{\mathbf{H}}(\mathbf{c}_{l,1} - \mathbf{c}_{l,2}) + \mathbf{v}_l\right\|^2 - \sum_{l=1}^{L}\|\mathbf{v}_l\|^2\,. \tag{21}$$

Let the $ML \times 1$ zero-mean complex jointly circular Gaussian vectors $\mathbf{g}$ and $\mathbf{v}$ be defined as

$$\mathbf{g} \triangleq \begin{bmatrix} \hat{\mathbf{H}}(\mathbf{c}_{1,1} - \mathbf{c}_{1,2}) \\ \hat{\mathbf{H}}(\mathbf{c}_{2,1} - \mathbf{c}_{2,2}) \\ \vdots \\ \hat{\mathbf{H}}(\mathbf{c}_{L,1} - \mathbf{c}_{L,2}) \end{bmatrix}, \tag{22a}$$

$$\mathbf{v} \triangleq \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_L \end{bmatrix}. \tag{22b}$$

Let

$$\mathbf{K}_g = \mathbf{E}\left[\mathbf{g}\mathbf{g}^H\right], \quad \mathbf{K}_v = \mathbf{E}\left[\mathbf{v}\mathbf{v}^H\right], \tag{23}$$

be the correlation matrices of $\mathbf{g}$ and $\mathbf{v}$, respectively, and

$$\mathbf{K}_{gv} = \mathbf{E}\left[\mathbf{g}\mathbf{v}^H\right] \tag{24}$$

the cross-correlation matrix.

Using the statistics of $\mathbf{H}$, $\mathbf{n}_1, \ldots, \mathbf{n}_L$, and $\mathbf{N}_p$, and the relation (12), we get

$$\begin{aligned}
\mathbf{K}_g &= \frac{\rho^2}{\beta}(\mathbf{C}_1 - \mathbf{C}_2)^T(\mathbf{C}_1 - \mathbf{C}_2)^* \\
&\quad \otimes [\beta\mathbf{K} + 2N_0\mathbf{I}_M], && (25a)\\
\mathbf{K}_v &= (1-\rho)^2\mathbf{C}_1^T\mathbf{C}_1^* \otimes \mathbf{K} \\
&\quad + 2N_0\left[\frac{\rho^2}{\beta}\mathbf{C}_1^T\mathbf{C}_1^* + \mathbf{I}_L\right] \otimes \mathbf{I}_M, && (25b)\\
\mathbf{K}_{gv} &= (\mathbf{C}_1 - \mathbf{C}_2)^T\mathbf{C}_1^* \otimes \Bigg[\rho(1-\rho)\mathbf{K} \\
&\quad - 2N_0\frac{\rho^2}{\beta}\mathbf{I}_M\Bigg], && (25c)
\end{aligned}$$

where $(\cdot)^T$ denotes transpose, $(\cdot)^*$ the complex conjugate, and $\otimes$ the Kronecker product of matrices.

The characteristic function (c.f.) of the decision variable is given by

$$\begin{aligned}
&\Psi_{\mathcal{D}}(\jmath\omega) \\
&= \mathbf{E}\left[e^{\jmath\omega\mathcal{D}}\right] \\
&= \frac{1}{\det\left(\mathbf{I}_{2ML} - \jmath\omega\begin{bmatrix}\mathbf{I}_{ML} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{ML}\end{bmatrix}\mathbf{K}_{\mathcal{D}}\right)}, && (26)
\end{aligned}$$

where $\mathbf{K}_{\mathcal{D}}$ is expressed as

$$\mathbf{K}_{\mathcal{D}} = \begin{bmatrix} (\mathbf{K}_{gv} + \mathbf{K}_v \\ \mathbf{K}_g + \mathbf{K}_{gv}^H) & \mathbf{K}_{gv} + \mathbf{K}_v \\ \mathbf{K}_{gv}^H + \mathbf{K}_v & \mathbf{K}_v \end{bmatrix}. \tag{27}$$

Substituting (27) in (26), we get

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{\det\begin{pmatrix} \mathbf{I}_{ML} - \jmath\omega(\mathbf{K}_g + \mathbf{K}_{gv} + \mathbf{K}_{gv}^H) \\ +\omega^2\Big((\mathbf{K}_{gv} + \mathbf{K}_v + \mathbf{K}_g + \mathbf{K}_{gv}^H)\mathbf{K}_v \\ -(\mathbf{K}_{gv} + \mathbf{K}_v)(\mathbf{I}_{ML} + \jmath\omega\mathbf{K}_v)^{-1} \\ \times(\mathbf{K}_{gv}^H + \mathbf{K}_v)(\mathbf{I}_{ML} + \jmath\omega\mathbf{K}_v)\Big) \end{pmatrix}}. \tag{28}$$

We consider the case when the transmitted codes are square orthogonal codes, implying $N = L$, which gives the condition

$$\mathbf{C}_i\mathbf{C}_i^H = \mathbf{C}_i^H\mathbf{C}_i = \alpha\mathbf{I}_N, \ \ \alpha > 0, \quad i = 1, 2. \tag{29}$$

On applying this condition, (25) can be simplified as

$$\begin{aligned}
\mathbf{K}_g &= \frac{\rho^2}{\beta}(2\alpha\mathbf{I}_N - \mathbf{C}_1^T\mathbf{C}_2^* - \mathbf{C}_2^T\mathbf{C}_1^*) \\
&\quad \otimes [\beta\mathbf{K} + 2N_0\mathbf{I}_M], && (30a)\\
\mathbf{K}_v &= (1-\rho)^2\alpha\mathbf{I}_N \otimes \mathbf{K} \\
&\quad + 2N_0\left(\frac{\rho^2}{\beta}\alpha + 1\right)\mathbf{I}_N \otimes \mathbf{I}_M, && (30b)\\
\mathbf{K}_{gv} &= (\alpha\mathbf{I}_N - \mathbf{C}_2^T\mathbf{C}_1^*) \otimes \Bigg[\rho(1-\rho)\mathbf{K} \\
&\quad - 2N_0\frac{\rho^2}{\beta}\mathbf{I}_M\Bigg]. && (30c)
\end{aligned}$$

Due to the condition given in (29), both $\mathbf{K}_{gv}^H$ and $\mathbf{K}_{gv}$ commute with $\mathbf{K}_v$. This simplifies the c.f., given by (26), as

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{\det\begin{pmatrix} \mathbf{I}_{ML} - \jmath\omega(\mathbf{K}_g + \mathbf{K}_{gv} + \mathbf{K}_{gv}^H) \\ +\omega^2\Big((\mathbf{K}_{gv} + \mathbf{K}_v + \mathbf{K}_g + \mathbf{K}_{gv}^H)\mathbf{K}_v \\ -(\mathbf{K}_{gv} + \mathbf{K}_v)(\mathbf{K}_{gv}^H + \mathbf{K}_v)\Big) \end{pmatrix}}. \tag{31}$$

Substitution of (30) in (31) results in

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{\det\begin{pmatrix} \mathbf{I}_{NM} - \jmath\omega\rho\boldsymbol{\Delta} \otimes \mathbf{K} + \omega^2(2N_0)^2 \\ \times\left[\frac{\rho^2}{\beta}\boldsymbol{\Delta} \otimes \mathbf{I}_M + \frac{\rho^2}{2N_0}\left(\frac{\alpha}{\beta} + 1\right)\boldsymbol{\Delta} \otimes \mathbf{K}\right] \end{pmatrix}}, \tag{32}$$

where $\boldsymbol{\Delta}$, the code difference matrix, is defined as

$$\begin{aligned}
\boldsymbol{\Delta} &\triangleq (\mathbf{C}_1 - \mathbf{C}_2)^T(\mathbf{C}_1 - \mathbf{C}_2)^* \\
&= 2\alpha\mathbf{I}_N - \mathbf{C}_2^T\mathbf{C}_1^* - \mathbf{C}_1^T\mathbf{C}_2^*. && (33)
\end{aligned}$$

Consider the case when $\boldsymbol{\Delta}$ is a scaled version of the identity matrix. If

$$\boldsymbol{\Delta} = \alpha\nu\mathbf{I}_N, \quad \nu > 0, \tag{34}$$

and

$$\begin{aligned}
\Gamma &= \frac{\alpha\Omega}{2N_0}, \\
\Gamma_p &= \frac{\beta\Omega}{2N_0}, && (35)
\end{aligned}$$

Γ denoting the *average SNR per diversity branch* and Γ_p the *average pilot power to noise power ratio* (APPNPR), then we can rewrite (32) as

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{\left\{\det\begin{pmatrix} \mathbf{I}_M - \jmath\omega(2N_0)\rho\nu\Gamma\frac{1}{\Omega}\mathbf{K} \\ +\omega^2(2N_0)^2\rho^2\frac{\Gamma}{\Gamma_p}\nu\left[\mathbf{I}_M + (\Gamma + \Gamma_p)\frac{1}{\Omega}\mathbf{K}\right] \end{pmatrix}\right\}^N}. \tag{36}$$

Let $\epsilon_1, \cdots, \epsilon_K$ denote the K distinct eigenvalues of $\mathbf{K}/\Omega$, such that ϵ_i has multiplicity q_i for $i = 1, \ldots, K$. Thus $q_1 + \cdots + q_K = M$, and, as per condition (9), $\sum_{i=1}^{K} q_i \epsilon_i = M$. Note that $\epsilon_1, \ldots, \epsilon_K$ are real and positive. The c.f. can now be rewritten as

$$\begin{aligned}
&\Psi_{\mathcal{D}}(\jmath\omega) \\
&= \frac{1}{\prod_{i=1}^{K} \left(\begin{array}{l} 1 - \jmath\omega(2N_0)\rho\nu\Gamma\epsilon_i \\ +\omega^2(2N_0)^2\rho^2 \frac{\Gamma}{\Gamma_p}\nu\left[1 + (\Gamma + \Gamma_p)\epsilon_i\right] \end{array} \right)^{Nq_i}} \\
&= \frac{1}{\prod_{i=1}^{K} \left[(1 - 2\jmath\omega N_0\rho\lambda_{i1})^{Nq_i}(1 - 2\jmath\omega N_0\rho\lambda_{i2})^{Nq_i}\right]}, \quad (37)
\end{aligned}$$

where

$$\begin{aligned}
\lambda_{i1} &= \frac{\Gamma\nu\epsilon_i + \sqrt{\Gamma^2\nu^2\epsilon_i^2 + 4\left(\frac{\Gamma}{\Gamma_p} + \left[1 + \frac{\Gamma}{\Gamma_p}\right]\Gamma\epsilon_i\right)\nu}}{2}, \\
\lambda_{i2} &= \frac{\Gamma\nu\epsilon_i - \sqrt{\Gamma^2\nu^2\epsilon_i^2 + 4\left(\frac{\Gamma}{\Gamma_p} + \left[1 + \frac{\Gamma}{\Gamma_p}\right]\Gamma\epsilon_i\right)\nu}}{2}.
\end{aligned} \quad (38)$$

Note that $\lambda_{i1} > 0$ while $\lambda_{i2} < 0$. The poles of $(\Psi_{\mathcal{D}}(z/(2N_0\rho)))/z$ which are on the left-half z-plane are λ_{i2}^{-1}, $i = 1, \ldots, K$. The PEP, which is the probability $\Pr(\mathcal{D} < 0)$, can be obtained from the c.f. using the residues at these poles. Applying the inversion theorem [7] followed by the Faa di Bruno's formula [8] as in [4], the PEP can be expressed in closed form as

$$\begin{aligned}
&P(\mathbf{C}_1 \to \mathbf{C}_2) \\
&= \sum_{j=1}^{K} \frac{(-\lambda_{j2})^{N(2M-q_j)}}{\left[\prod_{i=1}^{K}(\lambda_{i1} - \lambda_{j2})^{Nq_i}\right]\left[\prod_{\substack{i=1 \\ i\neq j}}^{K}(\lambda_{i2} - \lambda_{j2})^{Nq_i}\right]} \\
&\times \sum_{\substack{(l_1,\ldots,l_{Nq_j-1}) \\ 0\le l_1,\ldots,l_{Nq_j-1}\le Nq_j-1 \\ l_1+2l_2+\cdots+(Nq_j-1)l_{Nq_j-1}=Nq_j-1}} \prod_{m=1}^{Nq_j-1} \frac{1}{l_m!}\left[\frac{1}{m}\right. \\
&\qquad + \frac{N}{m}\sum_{i=1}^{K}\frac{q_i\lambda_{i1}^m}{(\lambda_{i1} - \lambda_{j2})^m} \\
&\qquad \left. + \frac{N}{m}\sum_{\substack{i=1 \\ i\neq j}}^{K}\frac{q_i\lambda_{i2}^m}{(\lambda_{i2} - \lambda_{j2})^m}\right]^{l_m}, \quad (39)
\end{aligned}$$

where $\{\lambda_{i1}\}$, $\{\lambda_{i2}\}$ are given by (38), and the summation is over all $(Nq_j - 1)$-tuples $(l_1, \ldots, l_{Nq_j-1})$ of integers in the range $[0, Nq_j - 1]$ satisfying $\sum_{m=1}^{Nq_j-1} ml_m = Nq_j - 1$. Thus (39) is a *closed-form expression for the PEP* in terms of the distinct eigenvalues $\epsilon_1, \ldots, \epsilon_K$ of $\mathbf{K}/\Omega$.

IV. Performance of Optimum Receiver

Let $\mathbf{h}$ and $\hat{\mathbf{h}}$ denote the $MN \times 1$ vectors given by

$$\mathbf{h} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \vdots \\ \mathbf{h}_N \end{bmatrix}, \quad \hat{\mathbf{h}} = \begin{bmatrix} \hat{\mathbf{h}}_1 \\ \hat{\mathbf{h}}_2 \\ \vdots \\ \hat{\mathbf{h}}_N \end{bmatrix}, \quad (40)$$

where $\mathbf{h}_1, \ldots, \mathbf{h}_N$ and $\hat{\mathbf{h}}_1, \ldots, \hat{\mathbf{h}}_N$ are the columns of $\mathbf{H}$ as in (2) and $\hat{\mathbf{H}}$ as in (6), respectively. The vector $\hat{\mathbf{h}}$ is thus an estimate of $\mathbf{h}$. The decision criterion for the optimum receiver is given by

$$\hat{\mathbf{C}}_1 = \arg\left\{ \max_{[\mathbf{X}_1,\cdots,\mathbf{X}_L]\in\mathcal{C}_{N,L}} f(\mathbf{r}|\hat{\mathbf{h}}, \mathbf{C}_1) \right\}, \quad (41)$$

where the $ML \times 1$ vector $\mathbf{r}$ is denoted as

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_L \end{bmatrix}. \quad (42)$$

The probability density function (p.d.f.) can be written as

$$\begin{aligned}
f(\mathbf{r}|\hat{\mathbf{h}}, \mathbf{C}) &= \frac{f(\mathbf{r}, \hat{\mathbf{h}}|\mathbf{C})}{f(\hat{\mathbf{h}})} \\
&= \frac{\int f(\mathbf{r}|\mathbf{h}, \mathbf{C}) f(\hat{\mathbf{h}}|\mathbf{h}) f(\mathbf{h})\, d\mathbf{h}}{f(\hat{\mathbf{h}})}. \quad (43)
\end{aligned}$$

Now the p.d.f. of $\mathbf{h}$ can be written in terms of the receiver correlation matrix $\mathbf{K}$ as

$$f(\mathbf{h}) = \frac{1}{(\det(\mathbf{K}))^N \pi^{MN}} \exp\left\{-\mathrm{tr}(\mathbf{H}^H\mathbf{K}^{-1}\mathbf{H})\right\}. \quad (44)$$

From (12), the p.d.f. of $\hat{\mathbf{h}}$, conditioned on $\mathbf{h}$, is given by

$$\begin{aligned}
f(\hat{\mathbf{h}}|\mathbf{h}) &= \frac{1}{(2\pi N_0)^{MN}\left[\det\left(\mathbf{Q}^T\mathbf{Q}^*\right)\right]^M} \\
&\quad \times \exp\left\{ -\frac{1}{2N_0}\mathrm{tr}\left[(\hat{\mathbf{H}} - \mathbf{H}\mathbf{C}_p\mathbf{Q})\right.\right. \\
&\quad \left.\left. \times (\mathbf{Q}^H\mathbf{Q})^{-1}(\hat{\mathbf{H}} - \mathbf{H}\mathbf{C}_p\mathbf{Q})^H\right]\right\}. \quad (45)
\end{aligned}$$

In addition, the p.d.f. of $\mathbf{r}$, conditioned on $\mathbf{h}$ and $\mathbf{C}$, is

$$f(\mathbf{r}|\mathbf{h},\mathbf{C}) = \frac{1}{(2\pi N_0)^{ML}} \times \exp\left\{ -\frac{1}{2N_0}\mathrm{tr}\left[(\mathbf{R}-\mathbf{HC}) \times (\mathbf{R}-\mathbf{HC})^H\right]\right\}. \quad (46)$$

Substituting (44), (45), (46) in (43) and using the decision rule (41), under the condition (8) and (29), we get

$$\hat{\mathbf{C}}_1 = \arg\left\{ \min_{[\ \mathbf{x}_1,\ \cdots,\ \mathbf{x}_L\ ]\in\mathcal{C}_{N,L}} \sum_{l=1}^{L}\left\|\mathbf{D}\mathbf{r}_l - \hat{\mathbf{H}}\mathbf{x}_l\right\|^2\right\}, \quad (47)$$

where $\mathbf{D}$ is given by

$$\mathbf{D} = \left(\mathbf{I}_M + \frac{1}{(\Gamma+\Gamma_p)}\Omega\mathbf{K}^{-1}\right)^{-1}. \quad (48)$$

Proceeding in a manner similar to the analysis for the suboptimum receiver, the PEP is given by

$$P(\mathbf{C}_1\to\mathbf{C}_2) = \Pr\left\{\sum_{l=1}^{L}\left\|\hat{\mathbf{H}}(\mathbf{c}_{l,1}-\mathbf{c}_{l,2})+\mathbf{v}_l\right\|^2 < \sum_{l=1}^{L}\|\mathbf{v}_l\|^2\right\}, \quad (49)$$

where $\mathbf{v}_l$ for the optimum receiver is expressed as

$$\mathbf{v}_l = \mathbf{D}(\mathbf{H}\mathbf{c}_{l,1}+\mathbf{n}_l) - \hat{\mathbf{H}}\mathbf{c}_{l,1}. \quad (50)$$

The decision variable is now

$$\mathcal{D} = \sum_{l=1}^{L}\left\|\hat{\mathbf{H}}(\mathbf{c}_{l,1}-\mathbf{c}_{l,2})+\mathbf{v}_l\right\|^2 - \sum_{l=1}^{L}\|\mathbf{v}_l\|^2. \quad (51)$$

The c.f. of the decision variable can now be written as

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{\det\begin{pmatrix}\mathbf{I}_{NM} - \jmath\omega\rho\mathbf{\Delta}\otimes\mathbf{KD} + \omega^2(2N_0)^2 \\ \times\left[\frac{\rho^2}{\beta}\mathbf{\Delta}\otimes\mathbf{D}^2 + \frac{\rho^2}{2N_0}\left(\frac{\alpha}{\beta}+1\right)\mathbf{\Delta}\otimes\mathbf{KD}^2\right]\end{pmatrix}}. \quad (52)$$

Again we consider the case when the code difference matrix $\mathbf{\Delta}$ is a scaled version of the identity matrix, satisfying (34). The c.f. can be simplified in terms of Γ and Γ_p given by (35) as

$$\Psi_{\mathcal{D}}(\jmath\omega) = \frac{1}{\left\{\det\begin{pmatrix}\mathbf{I}_M - \jmath\omega(2N_0)\rho\nu\Gamma\frac{1}{\Omega}\mathbf{KD} \\ +\omega^2(2N_0)^2\rho^2\frac{\Gamma}{\Gamma_p}\nu\left[\mathbf{D}^2+(\Gamma+\Gamma_p)\frac{1}{\Omega}\mathbf{KD}^2\right]\end{pmatrix}\right\}^N}. \quad (53)$$

From (48), the distinct eigenvalues of $\mathbf{D}$, given by $\epsilon_{D_1},\ldots,\epsilon_{D_K}$, can be expressed in terms of $\epsilon_1,\ldots,\epsilon_K$, the distinct eigenvalues of $\mathbf{K}/\Omega$, as

$$\epsilon_{D_i} = \frac{(\Gamma+\Gamma_p)\,\epsilon_i}{1+(\Gamma+\Gamma_p)\,\epsilon_i}. \quad (54)$$

We can express the c.f. of $\mathcal{D}$ in the form (37). Denoting λ_{i1} and λ_{i2} in the case of the optimum receiver as

$$\lambda_{i1} = \frac{\Gamma\nu\epsilon_i\epsilon_{D_i} + \sqrt{\Gamma^2\nu^2\epsilon_i^2\epsilon_{D_i}^2 + 4\left(\frac{\Gamma}{\Gamma_p}+\left[1+\frac{\Gamma}{\Gamma_p}\right]\Gamma\epsilon_i\right)\epsilon_{D_i}^2\nu}}{2},$$

$$\lambda_{i2} = \frac{\Gamma\nu\epsilon_i\epsilon_{D_i} - \sqrt{\Gamma^2\nu^2\epsilon_i^2\epsilon_{D_i}^2 + 4\left(\frac{\Gamma}{\Gamma_p}+\left[1+\frac{\Gamma}{\Gamma_p}\right]\Gamma\epsilon_i\right)\epsilon_{D_i}^2\nu}}{2}, \quad (55)$$

we can express the PEP by (39), with $\{\lambda_{i1}\}$, $\{\lambda_{i2}\}$ are given by (55).

V. Conclusion

We have obtained closed-form expressions for the PEP for suboptimum and optimum receivers when the code difference matrix is a scaled version of the identity matrix. As the SNR becomes large, Γ and Γ_p increase, and the values of $\{\lambda_{i1}\}$, $\{\lambda_{i2}\}$ in the two cases become close to each other, reducing the performance gap between the two receivers. The same happens when the off-diagonal elements of the normalized receive correlation matrix $\mathbf{K}/\Omega$ decrease in magnitude.

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