Averaging $Q(\|\mathbf{X}\|)$ for a Complex Gaussian Random Vector $\mathbf{X}$: A Novel Approach

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***Abstract*— In this paper, we compute $E[Q(\|\mathbf{X}\|]$, where $\mathbf{X}$ is an $n \times 1$ complex circularly Gaussian vector, $\|\mathbf{X}\|$ is the L^2 norm of $\mathbf{X}$ and $E[\ ]$ is the expectation operator. This is done by finding the characteristic function of the decision variable and subsequently applying the inversion formula to obtain a one dimensional real integral expression. This integral is then converted to a contour integral which is evaluated using a variant of the Cauchy's integral formula to obtain an expression for $E[Q(\|\mathbf{X}\|]$. We then provide some applications of the above result by obtaining expressions for error probabilities in fading channels.**

I. Introduction

The average of a Q-function expression is of interest in finding general expressions for the probability of symbol/bit error in slowly fading communication channels, where the argument of the Q-function is a function of a random variable with a well defined probability density function that depends on the kind of fading experienced by the channel [1]. Another application is in finding the bit error probability of the desired user in a fading channel when multiuser detection is being employed at the receiver [2].

The average of a Q-function whose argument is proportional to the square root of a non-central chi-squared random variable with $2n$ degrees of freedom is obtained by deriving a recursion relation [3]. However, this approach involves integrating the product of the Q-function and the chi-square probability density function. [2]. For the central chi-square distribution, an approach using Craig's formula [4] and the moment generating function (MGF) of the random variable is outlined in [1]. This method is quite complicated but gives the most general closed form expression when the Q-function argument is proportional to the square root of a Nakagami-m [5] random variable.

In this paper, we propose a simple method to average the Q-function whose argument is the L^2 norm of a non zero-mean complex circularly Gaussian vector using the characteristic function (CF) of a non-central chi-square distribution [5] and the inversion formula for the cumulative distribution function [6].

II. Problem Statement

Let $\mathbf{X}$ be an $n \times 1$ vector whose entries are complex circularly Gaussian random variables such that

$$\begin{aligned} E[\mathbf{X}] &= \mathbf{m}, \\ E[(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{\dagger}] &= 2\sigma^2\mathbf{I}_n, \end{aligned} \tag{1}$$

where $\{\dagger\}$ represents the complex conjugate-transpose operation and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Let $R = \|\mathbf{X}\|^2$, where $\|\mathbf{X}\|$ is the L^2 norm of the vector $\mathbf{X}$. Then R is non-central chi-square distributed with $2n$ degrees of freedom [5] and

$$E[Q(\|\mathbf{X}\|] = E[Q(\sqrt{R})]. \tag{2}$$

Let V be a Gaussian random variable with zero mean and unit variance. From [7], we obtain

$$\begin{aligned} E[Q(\sqrt{R})] &= E[P(V > \sqrt{R})] \\ &= \frac{1}{2}P(R - V^2 < 0) \\ &= \frac{1}{2}P(\Delta < 0), \end{aligned} \tag{3}$$

where $\Delta = R - V^2$. The characteristic function of Δ is given by

$$\begin{aligned} \Phi_\Delta(t) &= E[e^{jt\Delta}] = E[e^{jtR}]E[e^{-jtV^2}] \\ &= \Phi_R(t)\Phi_{V^2}(-t). \end{aligned} \tag{4}$$

Since V is Gaussian, V^2 chi-square distributed. Hence, we obtain [5]

$$\begin{aligned} \Phi_R(t) &= \frac{e^{\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t}}}{(1-2j\sigma^2 t)^n}, \\ \Phi_{V^2}(-t) &= \frac{1}{(1+2jt)^{\frac{1}{2}}}. \end{aligned} \tag{5}$$

Substituting $\Phi_R(t)$ and $\Phi_{V^2}(-t)$ from (5) in (4),

$$\Phi_\Delta(t) = \frac{e^{\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t}}}{(1-2j\sigma^2 t)^n (1+2jt)^{\frac{1}{2}}}. \tag{6}$$

Since

$$\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t} = -\frac{\|\mathbf{m}\|^2}{2\sigma^2} + \frac{\|\mathbf{m}\|^2}{2\sigma^2(1-2j\sigma^2 t)}, \tag{7}$$

(6) can be written as

$$\Phi_\Delta(t) = \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}} e^{\frac{\|\mathbf{m}\|^2}{2\sigma^2(1-2j\sigma^2 t)}}}{(1-2j\sigma^2 t)^n (1+2jt)^{\frac{1}{2}}}. \tag{8}$$

According to the inversion formula of Gil-Pelaez [6], the cumulative distribution function

$$
\begin{aligned}
F_\Delta(x) &= P(\Delta < x) \qquad (9)\\
&= \frac{1}{2} + \frac{1}{2\pi j}\int_0^\infty \frac{e^{jtx}\Phi_\Delta(-t) - e^{-jtx}\Phi_\Delta(t)}{t} dt.
\end{aligned}
$$

From (9), we have

$$
\begin{aligned}
P(\Delta < 0) &= \frac{1}{2} + \frac{1}{2\pi j}\int_0^\infty \frac{\Phi_\Delta(-t) - \Phi_\Delta(t)}{t} dt \qquad (10)\\
&= \frac{1}{2} + \frac{1}{2\pi j}\int_{-\infty}^\infty \frac{\Phi_\Delta(-t)}{t} dt.
\end{aligned}
$$

From (8) and (10),

$$
P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j}\int_{-\infty}^\infty \frac{e^{\frac{\|\mathbf{m}\|^2}{2\sigma^2(1+2j\sigma^2 t)}}}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^n} dt. \qquad (11)
$$

Expanding the exponential in the numerator of the integrand in (11) as a power series and interchanging the order of integration and summation, we obtain

$$
P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j}\sum_{p=0}^\infty \frac{1}{p!}\left(\frac{\|\mathbf{m}\|^2}{2\sigma^2}\right)^p \int_{-\infty}^\infty \frac{dt}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^{n+p}}. \qquad (12)
$$

It is easy to verify that

$$
\frac{1}{t(1+2j\sigma^2 t)^n} = \frac{1}{t} - 2j\sigma^2 \sum_{k=1}^n \frac{1}{(1+2j\sigma^2 t)^k}. \qquad (13)
$$

If we let

$$
I_n = \int_{-\infty}^\infty \frac{dt}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^n}, \qquad (14)
$$

(12) can be written as

$$
P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j}\sum_{p=0}^\infty \frac{1}{p!}\left(\frac{\|\mathbf{m}\|^2}{2\sigma^2}\right)^p I_{n+p} \qquad (15)
$$

From (13) and (14), we now have

$$
I_n = \int_{-\infty}^\infty \frac{1}{(1-2jt)^{\frac{1}{2}}}\left[\frac{1}{t} - 2j\sigma^2\sum_{k=1}^n \frac{1}{(1+2j\sigma^2 t)^k}\right] dt, \qquad (16)
$$

which, after changing the order of the integral and the summation can be written as

$$
I_n = \int_{-\infty}^\infty \frac{dt}{t(1-2jt)^{\frac{1}{2}}} - 2j\sigma^2 \sum_{k=1}^n \int_{-\infty}^\infty \frac{dt}{(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^k}.
$$

In the above, letting

$$
\begin{aligned}
J &= \int_{-\infty}^\infty \frac{dt}{t(1-2jt)^{\frac{1}{2}}}, \qquad (17)\\
J_k &= 2j\sigma^2 \int_{-\infty}^\infty \frac{dt}{(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^k}, \qquad (18)
\end{aligned}
$$

we can write (14) as

$$
I_n = J - \sum_{k=1}^n J_k. \qquad (19)
$$

In the next section, we first show that J and J_k can be reduced to simple real and contour integrals respectively and then solve them.

III. Solving for I_n

A. The Real Integral

In (17), through a change of variables (from t to $-t$), we obtain

$$
J = -\int_{-\infty}^\infty \frac{dt}{t(1+2jt)^{\frac{1}{2}}}. \qquad (20)
$$

Adding the expressions for J in (17) and (20),

$$
\begin{aligned}
2J &= \int_{-\infty}^\infty \frac{dt}{t(1-2jt)^{\frac{1}{2}}} - \int_{-\infty}^\infty \frac{dt}{t(1+2jt)^{\frac{1}{2}}} \qquad (21)\\
&= \int_{-\infty}^\infty \frac{1}{t}\left[\frac{(1+2jt)^{\frac{1}{2}} - (1-2jt)^{\frac{1}{2}}}{(1+4t^2)^{\frac{1}{2}}}\right] dt. \qquad (22)
\end{aligned}
$$

Multiplying the numerator and denominator of the integrand in (22) by $\left[(1+2jt)^{\frac{1}{2}} - (1-2jt)^{\frac{1}{2}}\right]$,

$$
J = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{t}\left[\frac{(1+2jt) - (1-2jt)}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}}\right\}}\right] dt. \qquad (23)
$$

Cancelling out all common factors,

$$
J = 2j\int_{-\infty}^\infty \frac{dt}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}}\right\}}. \qquad (24)
$$

In the above equation, we note that $(1-2j)^{\frac{1}{2}}$ is the complex conjugate of $(1+2j)^{\frac{1}{2}}$. Hence, the integrand in (24) is real as well as an even function of t. Thus, we get (see Appendix)

$$
\begin{aligned}
J &= 4j\int_0^\infty \frac{dt}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}}\right\}} \qquad (25)\\
&= j\pi. \qquad (26)
\end{aligned}
$$

B. The contour integral

The integral in (18) can be solved easily if it can be converted to a contour integral. Towards this end, we state the following Lemma [8].

Lemma 3.1: Let g(x) be a function of a real variable x such that $|g(x)|$ has a denominator different from zero for all real x and is of degree in excess of a unit higher than the degree of the numerator. Then

$$
\int_{-\infty}^\infty g(x)dx = \int_C g(z)dz, \qquad (27)
$$

where C is a semicircle in the complex upper half-plane whose diameter is the real-axis and the integration is in the anti-clockwise sense.

For the integrand in (18), $k \geq 1$ and the degree of the denominator is greater than that of the numerator by $k + \frac{1}{2}$. From Lemma 3.1, we get

$$\begin{aligned} J_k &= 2j\sigma^2 \int_C \frac{dz}{(1-2jz)^{\frac{1}{2}}(1+2j\sigma^2 z)^k} \quad (28) \\ &= (2j\sigma^2)^{1-k} \int_C \frac{dz}{(1-2jz)^{\frac{1}{2}}(z-\frac{j}{2\sigma^2})^k}. \end{aligned}$$

We now present a formula for finding the derivatives of an analytic function [8] and subsequently use it to evaluate I_k.

Lemma 3.2: If $g(z)$ is analytic in a domain D, then it has derivatives of all orders in D which are then also analytic functions in D. The value of the $(k-1)$th derivative at a point z_0 in D is given by the formula

$$g^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi j} \int_L \frac{g(z)}{(z-z_0)^k} \quad (k = 1, 2, \ldots); \quad (29)$$

where L is any simple closed path in D which encloses z_0 an whose full interior belongs to D; the curve is traversed in the counterclockwise sense and $g^{(0)}(z_0) = g(z_0)$, by definition.
The function

$$f(z) = \frac{1}{(1-2jz)^{\frac{1}{2}}} \quad (30)$$

is analytic in the upper half-plane and C is a closed path in it. Since (28) can be written as

$$J_k = (2j\sigma^2)^{1-k} \int_C \frac{f(z)}{(z-\frac{j}{2\sigma^2})^k} dz, \quad (31)$$

and the point $\frac{j}{2\sigma^2}$ lies within C, using Lemma 3.2, we obtain

$$J_k = \frac{2\pi j(2j\sigma^2)^{1-k}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[\frac{1}{(1-2jz)^{\frac{1}{2}}}\right]_{z=\frac{j}{2\sigma^2}} \quad (32)$$

which, after some simplification, yields

$$\begin{aligned} J_k &= \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}}, k = 1, \quad (33) \\ &= \frac{2\pi j\sigma}{(k-1)!} \frac{\frac{1}{2}\frac{3}{2}\cdots\frac{(2k-3)}{2}}{(1+\sigma^2)^{k-\frac{1}{2}}}, \quad 1 < k \leq n. \end{aligned}$$

Substituting the expressions obtained in (26) and (33) in (19), we get

$$\begin{aligned} I_n &= j\pi - \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}}\left[1 + \sum_{k=2}^{n} \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2k-3)}{2}}{(k-1)!(1+\sigma^2)^{k-1}}\right] \quad (34) \\ &= j\pi - \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}}\left[1 + \sum_{k=1}^{n-1} \frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2k-1)}{2}}{k!(1+\sigma^2)^{k}}\right] \\ &= 2\pi j\left[\frac{1}{2} - \frac{\sigma}{\sqrt{1+\sigma^2}} \sum_{k=0}^{n-1} \binom{2k}{k} \left\{\frac{1}{4(1+\sigma^2)}\right\}^k\right]. \end{aligned}$$

IV. Closed form expression for $E[Q(\|\mathbf{X}\|)]$

From (15) and (34),

$$P(\Delta < 0) = 1 - \frac{\sigma e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{\sqrt{1+\sigma^2}} \sum_{p=0}^{\infty} \sum_{k=0}^{n+p-1} \frac{1}{p!} \left(\frac{\|\mathbf{m}\|^2}{2\sigma^2}\right)^p \times \binom{2k}{k} \left\{\frac{1}{4(1+\sigma^2)}\right\}^k. \quad (35)$$

Let $\alpha = \frac{\|\mathbf{m}\|}{2\sigma^2}$ and $\beta = \frac{1}{1+\sigma^2}$. Then, changing the indices of summation,

$$P(\Delta < 0) = 1 - e^{-\alpha}\sqrt{1-\beta}(A+B) \quad (36)$$

where[1]

$$A = \sum_{p=0}^{\infty} \sum_{k=0}^{p} \binom{2k}{k} \left\{\frac{\beta}{4}\right\}^k \frac{\alpha^p}{p!}, \quad (37)$$

$$B = \sum_{p=0}^{\infty} \sum_{k=p}^{n+p-1} \binom{2k}{k} \left\{\frac{\beta}{4}\right\}^k \frac{\alpha^p}{p!}. \quad (38)$$

We define the *factorial function* [9] as

$$(\gamma)_q = \prod_{r=1}^{q} (\gamma + r - 1), \quad (\gamma)_0 = 1, \gamma \neq 0, \quad (39)$$

where q is a positive integer.

A. The B series

Since

$$\binom{2k}{k} = \frac{4^k \left(\frac{1}{2}\right)_k}{k!}, \quad (40)$$

we obtain

$$B = \sum_{p=0}^{\infty} \sum_{k=p}^{n+p-1} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \frac{\alpha^p}{p!}. \quad (41)$$

Changing the limits of summation in (41),

$$\begin{aligned} B &= \sum_{p=0}^{\infty} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_{k+p} \beta^{k+p}}{(k+p)!} \frac{\alpha^p}{p!} \quad (42) \\ &= \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \sum_{p=0}^{\infty} \frac{\left(\frac{1}{2}+k\right)_p}{(k+1)_p} \frac{(\alpha\beta)^p}{p!} \\ &= \sum_{k=0}^{n-1} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}+k; k+1; \alpha\beta\right), \end{aligned}$$

where ${}_1F_1(a;b;x)$ is the confluent hypergeometric function [9]. According to Kummer's formula for the confluent hypergeometric function,

$${}_1F_1(a;b;x) = e^x {}_1F_1(b-a;b;-x). \quad (43)$$

Using this result in (42), we obtain

$$B = \exp(\alpha\beta) \sum_{k=0}^{n-1} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}; k+1; -\alpha\beta\right). \quad (44)$$

[1]We assume that all the infinite series considered henceforth converge.

B. The A series

We rewrite (37) as

$$A = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!}\frac{\alpha^p}{p!} - \sum_{p=0}^{\infty}\sum_{k=p+1}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!}\frac{\alpha^p}{p!} \qquad (45)$$

In the above,

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!}\frac{\alpha^p}{p!} = \left(\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\right)\left(\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k \beta^k}{k!}\right). \qquad (46)$$

Since $|\beta| < 1$, the second sum on the right hand side of (46) is the binomial series, i.e.,

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} = (1-\beta)^{-\frac{1}{2}}\,. \qquad (47)$$

Thus,

$$A = \frac{e^{\alpha}}{\sqrt{1-\beta}} - S, \qquad (48)$$

where

$$\begin{aligned} S &= \sum_{p=0}^{\infty}\sum_{k=p+1}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!}\frac{\alpha^p}{p!} \qquad (49)\\ &= \sum_{p=0}^{\infty}\sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+p} \beta^{k+p}}{(k+p)!}\frac{\alpha^p}{p!}. \end{aligned}$$

Following the steps in (42), (49) can be written as

$$S = \sum_{k=1}^{\infty}\binom{2k}{k}\left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}+k; k+1; \alpha\beta\right). \qquad (50)$$

The above infinite series has a closed form expression [3]

$$S = \frac{2\exp(\frac{\alpha\beta}{2})}{\sqrt{1-\beta}} \qquad (51)$$

$$\times\left[\exp\left\{\frac{\alpha}{2}(1+\beta)\right\}Q_1(u,w) - \frac{1}{2}(1+\sqrt{1-\beta})I_0\left(\frac{\alpha\beta}{2}\right)\right], \qquad (52)$$

where

$$\begin{aligned} u &= \sqrt{\frac{\alpha}{2}(2-\beta) - \frac{2}{\beta}\sqrt{1-\beta}} \qquad (53)\\ w &= \sqrt{\frac{\alpha}{2}(2-\beta) + \frac{2}{\beta}\sqrt{1-\beta}}, \end{aligned}$$

and $Q_1(u, w)$ is the Marcum Q-function [5]. Substituting (51) in (48) gives us a closed form expression for A. Since we already have a compact expression for B in (44), replacing the infinite series for A and B in (35) by their respective closed form expressions, and noting from (2) and (3) that

$$E[Q(\|\mathbf{X}\|)] = \frac{1}{2}P(\Delta < 0), \qquad (54)$$

we obtain an exact expression for $E[Q(\|\mathbf{X}\|)]$[2].

Corollary: When $\mathbf{X}$ is zero mean, from (34) and (35), we obtain

$$\begin{aligned} P(\Delta<0) &= \frac{1}{2} + \frac{I_n}{2\pi j} \qquad (55)\\ &= 1 - \frac{\sigma}{\sqrt{1+\sigma^2}}\sum_{k=0}^{n-1}\binom{2k}{k}\left\{\frac{1}{4(1+\sigma^2)}\right\}^k. \end{aligned}$$

Substituting the above in (54) leads to the well known result [2]

$$E[Q(\|\mathbf{X}\|)] = \frac{1}{2}\left[1 - \frac{\sigma}{\sqrt{1+\sigma^2}}\sum_{k=0}^{n-1}\binom{2k}{k}\left\{\frac{1}{4(1+\sigma^2)}\right\}^k\right] \qquad (56)$$

V. Example: Average Probability of Error for Nakagami-m fading channels

If a random variable α is Nakagami-m distributed, the random variable $\gamma = \frac{\alpha^2\varepsilon_b}{N_0}$ has the probability density function [5]

$$p(\gamma) = \frac{m^m}{\Gamma(m)\bar{\gamma}}\gamma^{m-1}e^{-m\gamma/\bar{\gamma}}, \qquad (57)$$

where $\bar{\gamma} = \frac{E(\alpha^2)\varepsilon_b}{N_0}$. For fading channels, the average probability of error is given by

$$P_e = \int_0^{\infty} Q(a\sqrt{\gamma})p_{\gamma}(\gamma)d\gamma, \qquad (58)$$

where a is a constant that depends on the specific modulation/detection combination [1]. We note that γ has the same distribution as R (when $\mathbf{X}$ has zero mean) with $\sigma^2 = \frac{\bar{\gamma}}{2m}$ when m is an integer [5]. Thus, after accounting for the constant a in (58), the average probability of error for a Nakagami-m fading channel is obtained as

$$P_e = \frac{1}{2}\left[1 - \sqrt{\frac{a^2\bar{\gamma}}{2m+a^2\bar{\gamma}}}\sum_{k=0}^{m-1}\binom{2k}{k}\left\{\frac{2m}{4(2m+a^2\bar{\gamma})}\right\}^k\right] \qquad (59)$$

by substituting $\sigma^2 = \frac{a^2\bar{\gamma}}{2m}$ and replacing n by m in (56). We note that exactly the same result has been arrived at using a different approach in [1], equation (5.18).

Appendix

Let $1 + 2jt = re^{j\theta}$, where $r = (1+4t^2)^{\frac{1}{2}}$ and $\cos\theta = \frac{1}{r}$. Since

$$(1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}} = 2r^{\frac{1}{2}}\cos\frac{\theta}{2}, \qquad (60)$$

and

$$\begin{aligned} \cos\frac{\theta}{2} &= \sqrt{\frac{1}{2}(1+\cos\theta)} \qquad (61)\\ &= \sqrt{\frac{1}{2}\left(1+\frac{1}{r}\right)}, \end{aligned}$$

the integrand in (25)

$$\frac{1}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}}\right\}} = \frac{1}{2r\sqrt{\frac{1+r}{2}}}. \qquad (62)$$

[2]Lindsey, in [3], using a fundamentally different approach, has obtained a closed form expression in a completely different form.

Hence,

$$J = 4j\int_0^{\infty} \frac{dt}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}}+(1-2jt)^{\frac{1}{2}}\right\}} \quad (63)$$

$$= 2\sqrt{2}j\int_0^{\infty} \frac{dt}{(1+4t^2)^{\frac{1}{2}}\left\{1+(1+4t^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}}.$$

Substituting $t = \frac{\tan\phi}{2}$ in the above,

$$J = \sqrt{2}j\int_0^{\frac{\pi}{2}} \frac{\sec\phi}{\sqrt{1+\sec\phi}}d\phi. \quad (64)$$

Let $\sqrt{1+\sec\phi} = \sqrt{2}\sec\psi$. Then

$$\frac{\sec\phi}{\sqrt{1+\sec\phi}}d\phi = \sqrt{2}d\psi. \quad (65)$$

The integral in (64) now becomes

$$J = 2j\int_0^{\frac{\pi}{2}} d\psi \quad (66)$$

$$= j\pi.$$

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