THE MATRIX FRAMEWORK FOR COSINE MODULATED FILTER BANKS USING DCT-6

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ABSTRACT

The filter matrices of the analysis and synthesis filters of a perfect reconstruction cosine modulated filter bank are sparse, having either the diamond structure or the bidiagonal structure. If the filter matrices have bidiagonal form, then the analysis and synthesis polyphase matrices of the filter bank can be factorized in terms of certain types of nilpotent matrices. In this paper, we present the derivation of the filter matrix having a diamond structure for a discrete cosine transform type-6 (DCT-6) based cosine modulated filter bank. The conversion from the diamond structure to the bidiagonal structure is by simple, symmetric permutation and needs no additional hardware. The inverse of this permutation matrix is same as the matrix itself. The use of DCT-6 enables the design of linear phase perfect reconstruction (LPPR) modulated filter banks.

Keywords—Schuller-Smith framework, discrete cosine transform type-6, filter matrix, diamond matrix structure, bidiagonal matrix structure.

1. INTRODUCTION

The simple, compact matrix framework proposed by Schuller and Smith [5] offers a broader range of design flexibility for modulated filter banks. It also allows for simultaneous control of the overall system delay for a given filter length besides good filter quality, reconstruction quality and implementation structure. We designate this matrix framework as the Schuller-Smith framework. In [2]-[5] Schuller and Smith demonstrate the design of low-delay modulated filter banks having perfect reconstruction (PR) property in the mathematical sense. This formulation leads to the design of PR filter banks using nilpotent matrices [6], [7]. The resulting filter banks lack linear phase property. The design of LPPR cosine modulated filter banks using nilpotent matrices appears in [1]. This design employs DCT-6. In this paper, we show that the cosine modulated filter bank using DCT-6 actually leads to a sparse filter matrix having the diamond structure which we convert to the bidiagonal form through simple permutation without any need for additional hardware. This permutation matrix, being symmetric, has its inverse as the matrix itself. The bidiagonal structure enables the construction of a suitable nilpotent matrix polynomial to represent the filter matrix corresponding to a linear phase prototype FIR filter.

2. FORMULATION OF THE FILTER MATRIX

Consider an $M$-band $2M$-length cosine modulated filterbank using DCT-6. The filterbank may be viewed as processing the input in blocks. The output of the $k$th subband channel for the $m$th input block has the form

$$y_k(m) = \sum_{n=0}^{2M-1} x(mM+n)h(n) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} - \frac{M}{2} \right) \right); \forall m \in \mathbb{Z}, \quad (1)$$

where $x(\cdot)$ is the input sequence and $h(n)$ is the impulse response of the prototype filter. Every block of $M$ input samples, produces $M$ output samples in the variable $k$. For convenience, and without loss of generality, we can consider the analysis for $m=0$ and rewrite (1) as

$$y_k = \sum_{n=0}^{2M-1} x(n)h(n) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} - \frac{M}{2} \right) \right) \quad (2)$$

We shall formulate the analysis equation (2) as a folding operation followed by the cosine transform. This can be visualized as shown in the diagram in Fig.1. Fig.1-a shows a sequence containing $2M$-length block. This block consists of an input block of length $M$ followed by its replica. Fig.1-b shows the flipped version of the sequence in Fig.1-a. Fig.1-c depicts the flipped sequence in Fig.1-b shifted to the right by $\frac{M}{2}$ samples. We designate the part of this sequence from 0 to $\frac{M}{2} - 1$ as “term-1”. Fig.1-d represents the sequence in Fig.1-a shifted to the left by $\frac{M}{2}$ samples. The part of this sequence from 0 to $M - 1$ is “term-2”. Fig.1-e shows the block in Fig.1-a in reverse order. This is the flipped sequence in Fig.1-b shifted to the right by $2M - 1$ samples. The part of this sequence from 0 to $\frac{M}{2} - 1$ is “term-3”. The terms-1, 2 and 3 taken together cover the complete block of length $2M$.

Term-1 = The first $\frac{M}{2}$ samples of the $2M$-length block
Term-2 = The middle $M$ samples of the $2M$-length block
Term-3 = The last $\frac{M}{2}$ samples of the $2M$-length block
Thus, we shall rewrite (2) as

\[
y_k = \sum_{n=0}^{M-1} x \left( \frac{M}{2} - 1 - n \right) h \left( \frac{M}{2} - 1 - n \right) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right) + \sum_{n=0}^{M-1} x \left( n + \frac{M}{2} \right) h \left( n + \frac{M}{2} \right) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right) + \sum_{n=0}^{M-1} x \left( 2M - 1 - n \right) h \left( 2M - 1 - n \right) \cos \left( \frac{2\pi k}{2M-1} \left( \frac{M}{2} - 1 + n + \frac{1}{2} \right) \right). \tag{3}
\]

We can modify this expression as

\[
y_k = \sum_{n=0}^{M-1} x \left( \frac{M}{2} - 1 - n \right) h \left( \frac{M}{2} - 1 - n \right) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right) + \sum_{n=0}^{M-1} x \left( n + \frac{M}{2} \right) h \left( n + \frac{M}{2} \right) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right) + \sum_{n=0}^{M-1} x \left( n + \frac{M}{2} \right) h \left( n + \frac{M}{2} \right) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right) + \sum_{n=0}^{M-1} x \left( 5M - n - 2 \right) h \left( 5M - n - 2 \right) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right). \tag{4}
\]

This summation can be expressed in vector form as

\[
y_k = t_k \hat{x}, \tag{5}
\]

where \( y_k \) is the \( k \)th channel output corresponding to the input block, \( t_k \) is the \( k \)th row of the \( M \times M \) matrix of DCT-6 formed out of \( \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right) \), and \( \hat{x} \) is an \( M \times 1 \) vector with elements \( \hat{x}(n) \) defined as
\[
\begin{align*}
\hat{x}(0) &= x \left( \frac{M}{2} - 1 \right) h \left( \frac{M}{2} - 1 \right) + x \left( \frac{M}{2} \right) h \left( \frac{M}{2} \right), \\
\hat{x}(1) &= x \left( \frac{M}{2} - 2 \right) h \left( \frac{M}{2} - 2 \right) + x \left( \frac{M}{2} + 1 \right) h \left( \frac{M}{2} + 1 \right), \\
&\quad \vdots \quad \vdots \\
\hat{x} \left( \frac{3M}{2} - 1 \right) &= x(0) h(0) + x \left( M - 1 \right) h(M - 1), \\
\hat{x} \left( \frac{3M}{2} \right) &= x \left( 2M - 1 \right) h \left( 2M - 1 \right) + x(M) h(M), \\
&\quad \vdots \quad \vdots \\
\hat{x} \left( M - 1 \right) &= x \left( \frac{3M}{2} - 1 \right) h \left( \frac{3M}{2} - 1 \right) + x \left( \frac{3M}{2} \right) h \left( \frac{3M}{2} \right)
\end{align*}
\]

(6)

The generation of \( \hat{x}(n) \), \( n = 0 \cdots M - 1 \) and \( y_k, \; k = 0 \cdots M - 1 \) is shown in Fig.2.

We are viewing the filter bank as processing the input in blocks of length \( M \). We did the above formulation for the \( m \)th block-interval. Using this fact, we can write the general form of \( \hat{x}(n) \) as \( \hat{x}_m(n) \). The corresponding expressions for \( n = 0, \cdots, M - 1 \) are

\[
\begin{align*}
\hat{x}_m(0) &= x \left( mM + \frac{M}{2} - 1 \right) h \left( \frac{M}{2} - 1 \right) + x \left( mM + \frac{M}{2} \right) h \left( \frac{M}{2} \right), \\
\hat{x}_m(1) &= x \left( mM + \frac{M}{2} - 2 \right) h \left( \frac{M}{2} - 2 \right) + x \left( mM + \frac{M}{2} + 1 \right) h \left( \frac{M}{2} + 1 \right), \\
&\quad \vdots \quad \vdots \\
\hat{x}_m \left( \frac{3M}{2} - 1 \right) &= x(mM) h(0) + x \left( mM + M - 1 \right) h(M - 1), \\
\hat{x}_m \left( \frac{3M}{2} \right) &= x \left( mM + 2M - 1 \right) h \left( 2M - 1 \right) + x \left( mM + M \right) h(M), \\
&\quad \vdots \quad \vdots \\
\hat{x}_m \left( M - 1 \right) &= x \left( mM + \frac{3M}{2} - 1 \right) h \left( \frac{3M}{2} - 1 \right) + x \left( mM + \frac{3M}{2} \right) h \left( \frac{3M}{2} \right)
\end{align*}
\]

(7)

We shall view each \( \hat{x}_m(n) \), \( n = 0, \cdots, M - 1 \), as a sequence in the index \( m \). Thus, we can write the expression for the output of the \( k \)th filter corresponding to the \( m \)th input block as

\[
y_k(m) = \sum_{n=0}^{M-1} \hat{x}_m(n) \cos \left( \frac{2\pi k}{2M-1} \left( n + \frac{1}{2} \right) \right)
\]

(8)

This is the DCT-6 of \( \hat{x}_m(n) \). We have the \( m \)th input block as

\[
x = \begin{bmatrix} x_m(0) & x_m(1) & \cdots & x_m(M - 1) \end{bmatrix}^T
\]

(9)

and its z-transform as

\[
X(z) = [X_0(z) \quad X_1(z) \quad \cdots \quad X_{M-1}(z)]^T.
\]

(10)

The input vector to the DCT-6 matrix is

\[
\hat{x} = [\hat{x}_m(0) \quad \hat{x}_m(1) \quad \cdots \quad \hat{x}_m(M - 1)]^T
\]

(11)

having its z-transform as

\[
\hat{X}(z) = \begin{bmatrix} \hat{X}_0(z) & \hat{X}_1(z) & \cdots & \hat{X}_{M-1}(z) \end{bmatrix}^T.
\]

(12)

The output vector of the filter bank is

\[
y = [y_0(m) \quad y_1(m) \quad \cdots \quad y_{M-1}(m)]^T
\]

(13)

with its z-transform as

\[
Y(z) = [Y_0(z) \quad Y_1(z) \quad \cdots \quad Y_{M-1}(z)]^T.
\]

(14)

Our purpose is to find a folding matrix which converts \( X(z) \) into \( \hat{X}(z) \). Let us now find the expressions for the elements of \( \hat{X}(z) \). We define \( X(z) \) as the z-transform of the input sequence in the block index \( m \).

\[
\hat{X}_0(z) = X_{M-1}(z)h \left( \frac{M}{2} - 1 \right) + X_M(z)h \left( \frac{M}{2} \right).
\]

(15)
where
\[ X_{M-1}(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^{M-1} \]  
(16)
\[ X_M(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^M. \]  
(17)
\[ \hat{X}_1(z) = X_{M-2}(z)h\left(\frac{M}{2} - 2\right) + X_{M+1}(z)h\left(\frac{M}{2} + 1\right), \]  
(18)
where
\[ X_{M-2}(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^{M-2} \]  
(19)
\[ X_{M+1}(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^{M+1}. \]  
(20)

Similarly, we can obtain the expressions for the \(z\)-transforms of the remaining elements of \(X(z)\). We shall give the expressions for a few of them here.

\[ \hat{X}_{M-1}(z) = X_0(z)h(0) + X_{M-1}(z)h(M - 1), \]  
(21)
where
\[ X_0(z) = X(\frac{z}{M}) \]  
(22)
\[ X_{M-1}(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^{M-1}. \]  
(23)

Using the fact that \(x(mM + 2M - 1)\) and \(x(mM + M)\) are \(x(mM + M - 1)\) and \(x(mM + 0)\) shifted to the right by \(M\) samples each respectively, we can write \(\hat{X}_1(z)\) as

\[ \hat{X}_{M}(z) = X_{M-2}(z)h(2M - 1) + X_0(z)z^{-1}h(M), \]  
(24)
where
\[ X_0(z) = X(\frac{z}{M}) \]  
(25)
\[ X_{M-2}(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^{M-2}. \]  
(26)

Using the fact that \(x(mM + 2M - 2)\) and \(x(mM + M + 1)\) are \(x(mM + M - 2)\) and \(x(mM + 1)\) shifted to the right by \(M\) samples each respectively, we can write \(\hat{X}_{M+1}(z)\) as

\[ \hat{X}_{M+1}(z) = X_{M-2}(z)z^{-1}h(2M - 2) + X_1(z)z^{-1}h(M + 1), \]  
(27)
where
\[ X_1(z) = X(\frac{z}{M})\left(\frac{z}{M}\right) \]  
(28)
\[ X_{M-2}(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^{M-2}. \]  
(29)

In a similar way, we can find the rest of the expressions. For example, with \(x(mM + \frac{3M}{2})\) and \(x(mM + \frac{3M}{2} - 1)\) as \((mM + \frac{3M}{2})\) and \((mM + \frac{3M}{2} - 1)\) shifted to the right by \(M\) samples each respectively, we can write \(\hat{X}_{M-1}(z)\) as

\[ \hat{X}_{M-1}(z) = X_M(z)z^{-1}h\left(\frac{3M}{2}\right) + X_{M-1}(z)z^{-1}h\left(\frac{3M}{2} - 1\right), \]  
(30)

where
\[ X_{M-1}(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^{M-1} \]  
(31)
\[ X_M(z) = X(\frac{z}{M})\left(\frac{z}{M}\right)^M. \]  
(32)

Now we can relate \(X(z)\) with \(\hat{X}(z)\) in matrix form as

\[ \hat{X}(z) = F^{(D)}_a(z) \cdot X(z), \]  
(33)

where \(F^{(D)}_a(z)\) is a sparse matrix polynomial having diamond structure as

\[ F^{(D)}_a(z) = \begin{bmatrix} h(0) & h(M)z^{-1} & \ldots & \ldots & h(M - 2)z^{-1} \\ h(1) & h(M + 1)z^{-1} & \ldots & \ldots & h(M - 1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(M - 2) & h(M - 1) & \ldots & \ldots & h(2M - 2)z^{-1} \\ h(M - 1) & h(2M - 1)z^{-1} & \ldots & \ldots & h(2M - 2)z^{-1} \end{bmatrix} \]  
(34)

\(F^{(D)}_a(z)\) is the folding matrix. This is also known as the filter matrix. The general form of this matrix for a filter of length \(2ML\) is

\[ F^{(D)}_a(z) = \begin{bmatrix} P_0(z) & P_M(z)z^{-1} & \ldots & \ldots & P_{M-2}(z)z^{-1} \\ P_1(z) & P_{M+1}(z)z^{-1} & \ldots & \ldots & P_{M-1}(z)z^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{M-2}(z) & P_{M-1}(z)z^{-1} & \ldots & \ldots & P_{2M-2}(z)z^{-1} \\ P_{M-1}(z) & P_{2M-1}(z)z^{-1} & \ldots & \ldots & P_{2M-2}(z)z^{-1} \end{bmatrix} \]  
(35)

where
\[ P_\kappa(z) = \sum_{l=0}^{L-1} h(2lM + \kappa)z^{-2(l-1) - l}, \quad \kappa = 0, 1, \ldots, 2M-1. \]  
(36)
Now we can express $Y(z)$ as

$$Y(z) = T_0 F_{a}^{(D)}(z) X(z) = P_{a}(z) X(z), \quad (37)$$

where $T_0$ is the matrix of DCT-6 and $P_{a}(z) = T_0 F_{a}^{(D)}(z)$ is the polyphase matrix of the filter bank.

The diamond structure can be converted into a bidiagonal structure using permutation matrices without incurring any additional hardware cost. Let $J_R$ be a $M \times M$ permutation matrix having the form

$$J_R = \begin{bmatrix} J_{M/2} & 0 \\ 0 & J_{M/2} \end{bmatrix}, \quad (38)$$

where $J_{M/2}$ is an $M/2 \times M/2$ reversal matrix. We can convert the filter matrix $F_{a}^{(D)}(z)$, having diamond filter structure, into a bidiagonal form by pre-operating or post-operating it with $J_R$. Thus, we get the general form of the bidiagonal filter matrix $F_{a}^{(B)}(z)$ for the analyzer as $J_R$ is a symmetric permutation matrix. Therefore, its inverse is $J_R$ itself, avoiding the cost of matrix inversion at the synthesizer. The general form of $F_{a}^{(B)}(z)$ is

$$F_{a}^{(B)}(z) = \begin{bmatrix} P_{M-1}(z) & P_{2M-1}(z)z^{-1} \\ \vdots & \vdots \\ P_0(z) & P_{M}(z)z^{-1} \\ \vdots & \vdots \\ P_{M-1}(z) & P_{2M-1}(z)z^{-1} \\ P_{M/2}(z) & P_{M/2}(z)z^{-1} \end{bmatrix}, \quad (39)$$

where $P_{a}(z)$ is the same as in (36). Thus, we can rewrite the analysis polyphase matrix $P_{a}(z)$ as

$$P_{a}(z) = T_0 J_R F_{a}^{(B)}(z). \quad (40)$$

3. CONCLUSION

We have derived the filter matrix for a cosine modulated filter bank that uses DCT-6. The filter matrix has the diamond structure. The conversion between the diamond structure and the biaidiagonal structure has been carried out using an appropriate permutation matrix with trivial cost. The biaidiagonal structure can be represented in terms of nilpotent matrix polynomials. This result is useful in the design of LPPR cosine modulated filter banks using nilpotent matrices.

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