A Joint Performance Measure for the Decorrelating Multiuser Detector

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Abstract — We analyze the performance of the decorrelating multiuser detector for a wireless communication system. Instead of looking at the error probability for each user, we focus on the probability distribution of the number of correctly decoded users at the receiver and the average number of correctly decoded users under the conditions of equicorrelated signatures and perfect power control. The approach is motivated by the fact that the error events for different users are not independent. Numerical results show that when the number of users is large, instead of using a set of long signatures with a very low negative cross-correlation, we can use a set of shorter signatures with a reasonably high positive correlation without sacrificing the performance. It is also found that the average number of correctly decoded users increases with increase in signal-to-noise ratio and decreases when the cross-correlation between users becomes more negative or more positive.

I. INTRODUCTION

An important multiple access technique in wireless communications is code division multiple access (CDMA), which is a form of spread-spectrum [1, 2]. CDMA is used in various communication system standards, ranging from IS-95 to wideband CDMA [3, 4], and is expected to play a key role in the next generation mobile communication systems [5]. One salient feature of CDMA systems is the employment of multiuser detectors at the receivers to improve system performance. Hence the analysis of performance of multiuser detectors plays a very important role in the design of CDMA systems.

In the performance analysis of multiuser detectors, the error probability of each user is generally focussed on [6, 7]. However, the error events of the different users are not independent. Therefore, a joint performance measure, which accounts for all users together, can play a more effective role in system design. A joint performance measure is in tune with the basic philosophy behind multiuser detection, and can be used to determine, for example, the power levels of the users, the number of users that an environment can support, and the range of signature cross-correlation which can be used. In this paper,

we analyze the performance of the decorrelating detector, a key linear multiuser detector, by focussing on the probability distribution of the number of correctly decoded users at the receiver and the average number of correctly decoded users under the conditions of equicorrelated signatures and perfect power control.

II. THE CHANNEL MODEL

Consider the basic synchronous CDMA K -user channel model. The real-valued received signal $y(t)$, consisting of a sum of K antipodally modulated synchronous signature waveforms in additive white Gaussian noise, is given by

$$y(t) = \sum_{k=1}^K A_k b_k s_k(t) + \sigma n(t), \quad 0 \leq t < T, \quad (1)$$

where A_k , b_k , and $s_k(t)$ are the received amplitude, the bit transmitted, and the deterministic signature waveform assigned to the k th user, respectively. The signature waveform $s_k(t)$ is real-valued with support $[0, T)$ and unit energy, T being the bit interval or the inverse of the data rate. Thus

$$\int_0^T s_k^2(t) dt = 1. \quad (2)$$

The additive noise $n(t)$ is a zero-mean white Gaussian noise random process with unit power spectral density, implying

$$\mathbf{E}[n(t_1)n(t_2)] = \delta(t_1 - t_2),$$

$\mathbf{E}[\cdot]$ denoting the expectation operator, and $\delta(\cdot)$ the Dirac delta function. Note that $\sigma > 0$, $A_k > 0$, and $b_k \in \{-1, 1\}$.

The signal-to-noise ratio (SNR) of the k th user is given by the quantity A_k^2/σ^2 . The cross-correlation between the signature waveforms of the i th and j th users is defined as

$$\rho_{i,j} = \int_0^T s_i(t)s_j(t)dt, \quad (3)$$

and this gives a measure of the similarity between the waveforms $s_i(t)$ and $s_j(t)$. The *normalized cross-correlation matrix*, given by

$$\mathbf{R} = [\rho_{i,j}]_{i,j=1}^K, \quad (4)$$

is symmetric positive definite, implying that the signature waveforms $\{s_1(t), \dots, s_K(t)\}$ are linearly independent. Note from (2) that the matrix $\mathbf{R}$ has diagonal elements $\rho_{i,i}$, $i = 1, \dots, K$, equal to 1.

Multiuser detection is typically performed in the up-link, which can have code asynchronization. However, an asynchronous CDMA model with N users and frame length $2M + 1$ can be viewed as a synchronous model with $K = (2M + 1)N$ users [7]. Therefore, without loss of generality, we focus on the synchronous case.

The received waveform is converted into a discrete time process by passing it through a bank of matched filters, each matched to the signature waveform of a different user. The $K \times 1$ vector at the output of the K matched filters can be expressed as

$$\mathbf{y} = \mathbf{R}\mathbf{A}\mathbf{b} + \mathbf{n}, \quad (5)$$

where $\mathbf{R}$ is the normalized cross-correlation matrix given by (3),

$$\begin{aligned} \mathbf{y} &= [y_1, \dots, y_K]^T, \\ \mathbf{b} &= [b_1, \dots, b_K]^T, \\ \mathbf{A} &= \text{diag}(A_1, \dots, A_K), \end{aligned} \quad (6)$$

$(\cdot)^T$ denoting transpose, and $\mathbf{n}$ is a zero-mean Gaussian random vector with covariance matrix equal to

$$\mathbf{E}[\mathbf{n}\mathbf{n}^T] = \sigma^2\mathbf{R}. \quad (7)$$

Thus $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{R})$.

III. THE DECORRELATING DETECTOR

In the decorrelating detector, the bit b_k transmitted by the k th user is detected by the sign of the k th row of $\mathbf{R}^{-1}\mathbf{y}$, and is given by

$$\hat{b}_k = \text{sgn}((\mathbf{R}^{-1}\mathbf{y})_k), \quad (8)$$

where $\text{sgn}(\cdot)$ denotes the signum or sign function. From (6), we get

$$\mathbf{R}^{-1}\mathbf{y} = \mathbf{A}\mathbf{b} + \mathbf{R}^{-1}\mathbf{n}. \quad (9)$$

Define the random vector $\mathbf{w}$ as

$$\mathbf{w} = [w_1, \dots, w_K]^T \triangleq \mathbf{R}^{-1}\mathbf{n}. \quad (10)$$

We see from (7) and (10) that $\mathbf{w}$ is a zero-mean Gaussian random vector with covariance matrix $\sigma^2\mathbf{R}^{-1}$, that is,

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{R}^{-1}).$$

We can rewrite (9) as

$$(\mathbf{R}^{-1}\mathbf{y})_i = A_i b_i + w_i, \quad i = 1, \dots, K. \quad (11)$$

Consider the case of *equicorrelated signatures*, implying that the *cross-correlation between all users is uniform*, that is,

$$\rho_{i,j} = \rho \quad \forall i, j, \quad i \neq j. \quad (12)$$

Since $\det(\mathbf{R}) = (1 - \rho)^{K-1}(1 + (K - 1)\rho) > 0$, the range of ρ is

$$-\frac{1}{K-1} < \rho < 1. \quad (13)$$

It can be shown that the inverse of $\mathbf{R}$ has the form in which all the diagonal elements are a and off-diagonal elements are b , where

$$\begin{aligned} a &= \frac{1 + (K - 2)\rho}{(1 - \rho)(1 + (K - 1)\rho)}, \\ b &= -\frac{\rho}{(1 - \rho)(1 + (K - 1)\rho)}. \end{aligned} \quad (14)$$

Therefore we can write

$$\sigma^2\mathbf{R}^{-1} = \Omega \begin{bmatrix} 1 & \epsilon & \dots & \epsilon \\ \epsilon & 1 & \dots & \epsilon \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \epsilon & \cdot & \epsilon & 1 \end{bmatrix}, \quad (15)$$

with

$$\epsilon = -\frac{\rho}{1 + (K - 2)\rho}, \quad \Omega = \frac{\sigma^2(1 + (K - 2)\rho)}{(1 - \rho)(1 + (K - 1)\rho)}, \quad (16)$$

and

$$-\frac{1}{K-1} < \epsilon < 1. \quad (17)$$

We call ϵ the *modified cross-correlation*.

From (8) and (11), we can say that the i th user is correctly decoded when

$$\begin{aligned} w_i &> -A_i & \text{for } b_i = 1, \\ w_i &< -A_i & \text{for } b_i = -1. \end{aligned} \quad (18)$$

This can be rewritten as

$$b_i w_i > -A_i. \quad (19)$$

The *probability that the first n out of K users are correctly decoded*, which we denote as $P_{(n,K)}$, is given by

$$P_{(n,K)} = \Pr\left(b_1 w_1 > -A_1, \dots, b_n w_n > -A_n, \right. \\ \left. b_{n+1} w_{(n+1)} < -A_{(n+1)}, \dots, b_K w_K < -A_K\right) \quad (20)$$

Now consider the case when

$$0 \leq \epsilon < 1, \quad (21a)$$

which implies from (13) and (16) that

$$-\frac{1}{K-1} < \rho \leq 0. \quad (21b)$$

Note that (21a) is a subset of (17), and (21b) is a subset of (13). Thus when ϵ is nonnegative, the cross-correlation

ρ between users is nonpositive, and this enables us to express the random variables $w_1, \dots, w_K$ as [8]

$$w_i = (\sqrt{\epsilon} v_0 + \sqrt{1-\epsilon} v_i) \sqrt{\Omega}, \quad i = 1, 2, \dots, K, \quad (22)$$

where $v_0, v_1, \dots, v_K$ are independent and identically distributed Gaussian random variables each having a $\mathcal{N}(0, 1)$ distribution.

From (22), the condition (19) results in

$$b_i v_i > \frac{-\frac{A_i}{\sqrt{\Omega}} - \sqrt{\epsilon} b_i v_0}{\sqrt{1-\epsilon}}. \quad (23)$$

Note that $b_i v_i$ has a $\mathcal{N}(0, 1)$ distribution both for $b_i = 1$ and $b_i = -1$. Therefore, from (20), the probability that the first n out of K users are correctly decoded, conditioned on v_0 and $\mathbf{b}$, can be written as

$$P_{(n,K)}(v_0, \mathbf{b}) = \left[\prod_{i=1}^n Q\left(\frac{-\frac{A_i}{\sqrt{\Omega}} - \sqrt{\epsilon} b_i v_0}{\sqrt{1-\epsilon}}\right) \right] \times \left[\prod_{i=n+1}^K Q\left(\frac{\frac{A_i}{\sqrt{\Omega}} + \sqrt{\epsilon} b_i v_0}{\sqrt{1-\epsilon}}\right) \right], \quad (24)$$

where $Q(\cdot)$ denotes the Gaussian Q -function. We have used the property $Q(-x) = 1 - Q(x)$ in (24). Assuming independent and equiprobable bit transmission by each user, we average $P_{(n,K)}(v_0, \mathbf{b})$ over the probability density function of v_0 and over all possible bit sequences $\mathbf{b} \in \{-1, 1\}^K$, to obtain $P_{(n,K)}$, the probability that the first n out of K users are correctly decoded. This results in

$$P_{(n,K)} = \frac{1}{2^K} \sum_{\mathbf{b} \in \{-1, 1\}^K} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \times \left[\prod_{i=1}^n Q\left(\frac{-\frac{A_i}{\sqrt{\Omega}} - \sqrt{\epsilon} b_i v_0}{\sqrt{1-\epsilon}}\right) \right] \times \left[\prod_{i=n+1}^K Q\left(\frac{\frac{A_i}{\sqrt{\Omega}} + \sqrt{\epsilon} b_i v_0}{\sqrt{1-\epsilon}}\right) \right] dv_0. \quad (25)$$

The probability of error in the system of K users is given by

$$1 - P_{(K,K)} = 1 - \frac{1}{2^K} \sum_{\mathbf{b} \in \{-1, 1\}^K} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \times \left[\prod_{i=1}^K Q\left(\frac{-\frac{A_i}{\sqrt{\Omega}} - \sqrt{\epsilon} b_i v_0}{\sqrt{1-\epsilon}}\right) \right] dv_0 \quad (26)$$

Now let us look at the probability that n out of K users are correctly decoded, which we denote as $p(n, K)$. To characterize $p(n, K)$, we introduce the set $\mathcal{C}_{n,K}$, which denotes the set of those permutations of $(1, \dots, K)$ such that the first n elements are an n -combination of

$(1, \dots, K)$ in ascending order and the remaining $K - n$ elements are its complement in ascending order. Thus if $(c_1, \dots, c_n; c_{n+1}, \dots, c_K) \in \mathcal{C}_{n,K}$, then $c_1, \dots, c_n$ is an n -combination of $(1, \dots, K)$ with $c_1 < \dots < c_n$, and $c_{n+1}, \dots, c_K$ is the complement with $c_{n+1} < \dots < c_K$. For example, for $K = 4, n = 2$, we get

$$\mathcal{C}_{2,4} = \{(1, 2; 3, 4), (1, 3; 2, 4), (1, 4; 2, 3), (2, 3; 1, 4), (2, 4; 1, 3), (3, 4; 1, 2)\}.$$

Thus each $\mathcal{C}_{n,K}$ has $\binom{K}{n}$ K -tuples as its elements. The probability $p(n, K)$ can now be expressed as

$$p(n, K) = \sum_{(c_1, \dots, c_K) \in \mathcal{C}_{n,K}} \frac{1}{2^K} \sum_{\mathbf{b} \in \{-1, 1\}^K} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \times \left[\prod_{i=1}^n Q\left(\frac{-\frac{A_{c_i}}{\sqrt{\Omega}} - \sqrt{\epsilon} b_{c_i} v_0}{\sqrt{1-\epsilon}}\right) \right] \times \left[\prod_{i=n+1}^K Q\left(\frac{\frac{A_{c_i}}{\sqrt{\Omega}} + \sqrt{\epsilon} b_{c_i} v_0}{\sqrt{1-\epsilon}}\right) \right] dv_0. \quad (27)$$

Define quantities α_{c_i} , β_{c_i} , and γ as

$$\alpha_{c_i} \triangleq \begin{cases} -\frac{A_{c_i}}{\sqrt{\Omega(1-\epsilon)}}, & i = 1, \dots, n, \\ \frac{A_{c_i}}{\sqrt{\Omega(1-\epsilon)}}, & i = n+1, \dots, K, \end{cases} \quad (28a)$$

$$\beta_{c_i} \triangleq \begin{cases} -b_{c_i}, & i = 1, \dots, n, \\ b_{c_i}, & i = n+1, \dots, K, \end{cases} \quad (28b)$$

$$\gamma \triangleq \sqrt{\frac{\epsilon}{1-\epsilon}}. \quad (28c)$$

Using (28), we can rewrite (27) as

$$p(n, K) = \sum_{(c_1, \dots, c_K) \in \mathcal{C}_{n,K}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \times \sum_{\beta_{c_1}, \dots, \beta_{c_K} \in \{-1, 1\}} \left[\prod_{i=1}^K \frac{Q(\alpha_{c_i} + \beta_{c_i} \gamma v_0)}{2} \right] dv_0 \quad (29)$$

The sum of products of Gaussian Q -function terms in (29) can be expressed as

$$\begin{aligned} & \sum_{\beta_{c_1}, \dots, \beta_{c_K} \in \{-1, 1\}} \left[\prod_{i=1}^K \frac{Q(\alpha_{c_i} + \beta_{c_i} \gamma v_0)}{2} \right] \\ &= \sum_{\beta_{c_1}, \dots, \beta_{c_{K-1}} \in \{-1, 1\}} \left[\prod_{i=1}^{K-1} \frac{Q(\alpha_{c_i} + \beta_{c_i} \gamma v_0)}{2} \right] \\ & \quad \times \left[\frac{Q(\alpha_{c_K} + \gamma v_0) + Q(\alpha_{c_K} - \gamma v_0)}{2} \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{\beta_{c_1}, \dots, \beta_{c_{K-2}} \in \{-1, 1\}} \left[\prod_{i=1}^{K-2} \frac{Q(\alpha_{c_i} + \beta_{c_i} \gamma v_0)}{2} \right] \\
&\quad \times \left[\frac{Q(\alpha_{c_{K-1}} + \gamma v_0) + Q(\alpha_{c_{K-1}} - \gamma v_0)}{2} \right] \\
&\quad \times \left[\frac{Q(\alpha_{c_K} + \gamma v_0) + Q(\alpha_{c_K} - \gamma v_0)}{2} \right]. \quad (30)
\end{aligned}$$

Proceeding along the line of (30), we get

$$\begin{aligned}
&\sum_{\beta_{c_1}, \dots, \beta_{c_K} \in \{-1, 1\}} \left[\prod_{i=1}^K \frac{Q(\alpha_{c_i} + \beta_{c_i} \gamma v_0)}{2} \right] \\
&= \prod_{i=1}^K \frac{Q(\alpha_{c_i} + \gamma v_0) + Q(\alpha_{c_i} - \gamma v_0)}{2}. \quad (31)
\end{aligned}$$

Substituting (31) in (29), and expressing the result in terms of A_{c_i} , b_{c_i} , Ω , and ϵ using (28), we obtain

$$\begin{aligned}
p(n, K) &= \sum_{(c_1, \dots, c_K) \in \mathcal{C}_{n, K}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \\
&\quad \times \left[\prod_{i=1}^n \frac{Q\left(\frac{-\frac{A_{c_i}}{\sqrt{\Omega}} - \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right) + Q\left(\frac{-\frac{A_{c_i}}{\sqrt{\Omega}} + \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right)}{2} \right] \\
&\quad \times \left[\prod_{i=n+1}^K \frac{Q\left(\frac{\frac{A_{c_i}}{\sqrt{\Omega}} + \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right) + Q\left(\frac{\frac{A_{c_i}}{\sqrt{\Omega}} - \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right)}{2} \right] dv_0. \quad (32)
\end{aligned}$$

Consider further the case of *perfect power control*, when the *received signals of all users have equal power*. This implies

$$A_1 = A_2 = \dots = A_K = A. \quad (33)$$

Under the condition (33), we get from (32) the expression

$$\begin{aligned}
p(n, K) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \\
&\quad \times \binom{K}{n} [g(v_0)]^n [1 - g(v_0)]^{K-n} dv_0, \quad (34a)
\end{aligned}$$

where

$$\begin{aligned}
g(v_0) &= \frac{1}{2} \left[Q\left(\frac{-\frac{A}{\sqrt{\Omega}} - \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right) \right. \\
&\quad \left. + Q\left(\frac{-\frac{A}{\sqrt{\Omega}} + \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right) \right]. \quad (34b)
\end{aligned}$$

From (34), the *average number of correctly decoded users in the system*, which we denote as n_{av} , is given by

$$\begin{aligned}
n_{av} &= \sum_{n=0}^K n p(n, K) \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \\
&\quad \times \left[\sum_{n=0}^K n \binom{K}{n} [g(v_0)]^n [1 - g(v_0)]^{K-n} \right] dv_0 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} [K g(v_0)] dv_0 \\
&= \frac{K}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} \left[Q\left(\frac{-\frac{A}{\sqrt{\Omega}} - \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right) \right. \\
&\quad \left. + Q\left(\frac{-\frac{A}{\sqrt{\Omega}} + \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}}\right) \right] dv_0. \quad (35)
\end{aligned}$$

Using the Taylor's series expansion of the Gaussian Q -function, we can express $g(v_0)$ as

$$\begin{aligned}
g(v_0) &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{2^m m! (2m+1)} \\
&\quad \times \frac{1}{2} \left[\left(\frac{\frac{A}{\sqrt{\Omega}} + \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}} \right)^{2m+1} \right. \\
&\quad \left. + \left(\frac{\frac{A}{\sqrt{\Omega}} - \sqrt{\epsilon} v_0}{\sqrt{1-\epsilon}} \right)^{2m+1} \right]. \quad (36)
\end{aligned}$$

Observing that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} v_0^{2l} dv_0 &= \frac{(2l)!}{2^l l!}, \\
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v_0^2}{2}} v_0^{2l+1} dv_0 &= 0, \quad l = 0, 1, 2, \dots, \quad (37)
\end{aligned}$$

we use the binomial expansion formula in (36) and obtain from (35)

$$\begin{aligned}
n_{av} &= \frac{K}{2} - \frac{K}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{2^m m! (2m+1)} \\
&\quad \times \sum_{l=0}^m \binom{2m+1}{2l} \frac{(2l)!}{2^l l!} \left(\frac{\epsilon}{1-\epsilon} \right)^l \\
&\quad \times \left(\frac{A}{\sqrt{\Omega(1-\epsilon)}} \right)^{2m+1-2l}. \quad (38)
\end{aligned}$$

Substituting (16) in (38), we get the series expression

$$n_{av} = \frac{K}{2} - \frac{K}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-1)^{m-1} (2m)!}{2^m m!}$$

$$\begin{aligned} & \times \sum_{l=0}^m \frac{1}{2^l l! (2m+1-2l)!} \\ & \times \left(-\frac{\rho}{1+(K-1)\rho} \right)^l \\ & \times \left(\frac{A^2}{\sigma^2} (1-\rho) \right)^{m-l+\frac{1}{2}}. \end{aligned} \quad (39)$$

Although we started with the assumption that $\rho \leq 0$ for the expression (22) to make sense, it can be shown that the integral for $p(n, K)$ in (34a), the integral for n_{av} in (35), and the series expression for n_{av} in (39) are valid for any ρ satisfying (13). The reason for this is that when ρ is such that the argument of the Q -function is complex, a series expansion of the Q -function along with subsequent integration in (34a) results in a cancellation of all imaginary terms, leaving only the real terms which contribute to the probability $p(n, K)$.

A. Asymptotic Formulae

When the SNR A^2/σ^2 is large so as to satisfy $(A^2/\sigma^2)(1-\rho) \gg 1$, we can consider only the $l = 0$ term in the summation over l in (39), which results in

$$\begin{aligned} n_{av} \Big|_{\frac{A^2}{\sigma^2} \gg \frac{1}{1-\rho}} & \approx \frac{K}{2} - \frac{K}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}}{2^m m! (2m+1)} \\ & \times \left(\frac{A}{\sigma} \sqrt{1-\rho} \right)^{2m+1}. \end{aligned} \quad (40)$$

Again, from the Taylor's series expansion of the Gaussian Q -function, this can be rewritten as

$$n_{av} \Big|_{\frac{A^2}{\sigma^2} \gg \frac{1}{1-\rho}} \approx K \left[1 - Q \left(\frac{A}{\sigma} \sqrt{1-\rho} \right) \right]. \quad (41)$$

On the other hand, when the SNR A^2/σ^2 is small, we have the condition $A^2/\sigma^2 \ll 1$, which also implies $(A^2/\sigma^2)(1-\rho) \ll 1$. We can now consider only the $l = m$ term in the summation over l in (39). This results in

$$\begin{aligned} n_{av} \Big|_{\frac{A^2}{\sigma^2} \ll 1} & \approx \frac{K}{2} + \frac{K}{\sqrt{2\pi}} \left(\frac{A}{\sigma} \sqrt{1-\rho} \right) \\ & \times \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} \left(\frac{\rho}{4[1+(K-1)\rho]} \right)^m. \end{aligned} \quad (42)$$

We can rewrite (42) as

$$\begin{aligned} & n_{av} \Big|_{\frac{A^2}{\sigma^2} \ll 1} \\ & \approx K \left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \left(\frac{A}{\sigma} \sqrt{1-\rho} \right) \right. \\ & \quad \left. \times {}_2F_1 \left(\frac{1}{2}, 1; 1; \frac{\rho}{1+(K-1)\rho} \right) \right], \end{aligned} \quad (43)$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ denotes the Gaussian hypergeometric function.

IV. NUMERICAL RESULTS

In Fig. 1, the probability distribution of the number of correctly decoded users, computed using (34), is plotted at various values of cross-correlation ρ and constant SNR A^2/σ^2 in a system with $K = 20$ active users using the decorrelating detector. Negative values of ρ have been chosen. For example, a code consisting of 20 Gold sequences of length 31 has a uniform cross-correlation of $\rho = -1/31 = -0.0323$ between any pair of sequences (solid line in Fig. 1). We find that as ρ becomes more negative and the corresponding modified cross-correlation $\epsilon = -\rho/(1+(K-2)\rho)$ becomes more positive, the probability of higher number of users being correctly decoded decreases.

Fig. 2 shows plots of the average number n_{av} of correctly decoded users, obtained by computation using (35) as well as by computer simulation, versus the modified cross-correlation ϵ with different values of SNR in a system with $K = 20$ active users. We find that the simulation results match with the computed results. Simulations were performed with 10,000 runs for each value of n_{av} , and the maximum relative error

$$\left| \frac{(n_{av} \text{ obtained from simulation} - n_{av} \text{ obtained from computation})}{n_{av} \text{ obtained from computation}} \right|$$

over all the values of n_{av} obtained is 2.42%. It is found that for a given SNR, n_{av} is maximum for $\epsilon = 0$ which also implies $\rho = 0$. Therefore the performance is the best for orthogonal codes. As ϵ becomes more positive or more negative, n_{av} decreases. For a given ϵ , n_{av} increases with increase in SNR. Note that for $K = 20$, ϵ lies in the interval $(-1/19, 1)$. When $\epsilon = -0.05$ in Fig. 2, we have $\rho = 0.5$, while when $\epsilon = 0.5$, $\rho = -0.05$.

In Fig. 3, both computed and simulated values of n_{av} versus SNR are plotted with different values of ϵ in a system with $K = 20$ active users. The simulation results are found to match with the computed results. Simulations were again performed with 10,000 runs for each value of n_{av} , and the maximum relative error over all the values of n_{av} obtained is 1.86%.

An interesting finding from Figs. 2 and 3 is that n_{av} for some $\rho = \rho_0$ is the same as that for $\rho = -\rho_0/(1+(K-2)\rho_0)$. For ρ in the range $-1/(K-1) < \rho < 1$, a positive ρ implies a negative $-\rho/(1+(K-2)\rho)$ and vice versa. Thus, from Fig. 3, we find that when $K = 20$, the n_{av} for $\rho = 1/13 = 0.0769$ is the same as that for $\rho = -1/31 = -0.0323$, and the n_{av} for $\rho = 1/2 = 0.5$ is the same as that for $\rho = -1/20 = -0.05$. For a given K ,

Figure 1: Probability distribution of number of correctly decoded users at a constant SNR of 0 dB for a 20-user system

a signature set with a negative cross-correlation of, say, $-1/K$ will perform as well as a set with a positive cross-correlation of $1/2$. Therefore, when K is large, we need not use a set of long signatures with a very low negative cross-correlation; instead we can use a set of shorter signatures with a reasonably high positive correlation without sacrificing the performance.

We also find from Fig. 3 that an increase in SNR increases n_{av} . It can also be seen that as SNR increases, the average number of correctly decoded users asymptotically reaches the number K of active users in the system.

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Figure 2: Average number of correctly decoded users versus modified cross-correlation for a 20-user system

Figure 3: Average number of correctly decoded users versus SNR for a 20-user system